

Superfluid helium II as the QCD vacuum

Ariel Zhitnitsky

Department of Physics & Astronomy, University of British Columbia, Vancouver, B.C. V6T 1Z1, Canada

We study the winding number susceptibility $\langle I^2 \rangle$ in superfluid system and the topological susceptibility $\langle Q^2 \rangle$ in QCD. We argue that both correlation functions exhibit similar structures, including the generation of the contact terms. We discuss the nature of the contact term in superfluid system and argue that it has exactly the same origin as in QCD, and it is related to the long distance physics which cannot be associated with conventional microscopical degrees of freedom such as phonons and rotons. We emphasize that the conceptual similarities between superfluid system and QCD may lead, hopefully, to a deeper understanding of the topological features of a superfluid system as well as the QCD vacuum.

I. INTRODUCTION. MOTIVATION.

The main goal of this work is to present few arguments suggesting that a superfluid system has a number of features which are normally attributed to the QCD vacuum. In other words, while QCD is a system with a gap, it still exhibits some phenomena which are typically present in the systems with long range order such as superfluid liquid.

The basic objects of our study are the winding number susceptibility $\langle I^2 \rangle$ in superfluid system and the topological susceptibility $\langle Q^2 \rangle$ in QCD. The reason for our focus on these correlation functions is that the superfluid density n_s can be expressed in terms of the correlation function $\langle I^2 \rangle$, while the vacuum energy in QCD is explicitly expressed in terms of $\langle Q^2 \rangle$. Furthermore, the topological susceptibility $\langle Q^2 \rangle$ plays the crucial role in resolution of the celebrated $U(1)$ problem in QCD. Computation of such type of correlation functions is very hard technical problem which includes the dynamics of the non-local winding number operator as well as the dynamics of microscopical degrees of freedom carrying the vorticity. Fortunately, the topological susceptibility in QCD has been extensively studied in strongly coupled QCD and other gauge field theories with nontrivial topological features. The experience from these QCD studies may give us a hint about the behaviour of the winding number susceptibility $\langle I^2 \rangle$ in superfluid systems. Such an analogy (applying in the opposite way, from a superfluid system to QCD) may provide us with some new ideas on the nature of the phase transition in strongly coupled gauge theories when $\langle Q^2 \rangle$ experience some drastic changes according to the lattice studies.

We argue that these correlation functions demonstrate very similar features. In particular, they both exhibit the contact terms which are originated from the long distance dynamics not associated with any microscopical local propagating degrees of freedom such as well-studied in superfluidity phonons or rotons. This contact term is known to play a key role in the resolution of the $U(1)$ problem in QCD. We elaborate on properties of a similar contact term in superfluid systems.

The history of physics has a long list of examples when the conceptual similarity between particle physics and

condensed matter systems benefits both fields. In the present work we hope to extend this long list by adding one more example where the topological susceptibility $\langle Q^2 \rangle$ in QCD and correlation function $\langle I^2 \rangle$ in superfluid systems both exhibit similar and very unexpected properties.

Our presentation is organized as follows. In next few sections II, III we introduce our notations and definitions related to the topological properties of a superfluid liquid. We also argue that conventional Landau criterion cannot be used as a criterion for superfluidity. Rather a different criterion for superfluidity has to be used, and it should be formulated in terms of the winding number, rather than in terms of phonon- roton dispersion relation. The corresponding arguments will be reviewed in section IV.

In section V we introduce auxiliary gauge field to describe the vortices and circulation in superfluid systems, similar to vector gauge potential in $E\&M$ theory. In sections VI, VII we express the partition function and the winding number susceptibility $\langle I^2 \rangle$ in terms of these auxiliary topological gauge fields. Finally, in section VIII we elaborate on a number of similarities and differences between our computations of $\langle I^2 \rangle$ for superfluid liquid and the computations of $\langle Q^2 \rangle$ in QCD. We formulate the main lessons of our analysis in concluding section IX. We also speculate there on possible relevance of our studies for understanding the nature of the observed cosmological dark energy, as the vacuum energy in the system is directly related to the contact term in $\langle Q^2 \rangle$ in QCD (and $\langle I^2 \rangle$ in a superfluid system) which is the main object of our studies in the present work.

II. WINDING NUMBER AND ITS PROPERTIES

In what follows we specifically discuss a bosonic liquid such as ${}^4\text{He}$ to avoid any additional complications related to the fermionic structure of ${}^3\text{He}$ and its additional topological structures. One should comment that it is normally assumed the superfluid velocity is curl-free in superfluid ${}^4\text{He}$, i.e. $\vec{\nabla} \times \vec{v}_s = 0$. However, on a large scale the motion of superfluid ${}^4\text{He}$ is not really irrotational even when the density of the normal component

is very small. In fact, the corresponding circulation is quantized,

$$\oint \vec{v}_s \cdot d\vec{r} = \int d\vec{S} \cdot (\vec{\nabla} \times \vec{v}_s) = \frac{2\pi l \hbar}{m} = l \kappa_0, \quad (1)$$

where $\kappa_0 \equiv \frac{2\pi \hbar}{m}$ is unit flux of circulation and l is integer. It is important to emphasize that the quasiparticles, the phonons and rotons do not transfer energy directly to and from the superfluid component. Nevertheless, the interaction between the two components can be observed indirectly as the quasiparticles can scatter off the vortex lines. Eventual manifestation of this scattering is mutual friction between the components.

The key observation for our present discussions can be explained as follows. We want to express the superfluid density n_S in terms of specific correlation function $\langle I^2 \rangle$ formulated in terms of the winding number I as explained below section IV. The relevant for our discussions object is defined as follows [1–3],

$$I = \oint_{\Gamma} \vec{\nabla} \alpha \cdot d\vec{l} = 2\pi l, \quad \vec{I} \equiv \int_{\mathbb{M}} d^3x \vec{\nabla} \alpha. \quad (2)$$

In formula (2) the function $\alpha(x)$ describes the Nambu-Goldstone degree of freedom which itself is the phase of a scalar complex field $\Phi = \sqrt{n} \exp(i\alpha)$. A manifold \mathbb{M} in definition (2) is assumed to have at least one non-contractable path Γ such that there is at least one non-trivial mapping $\pi_1[U(1)] = \mathbb{Z}$ between $U(1)$ phase α and path Γ . In case of a 3-torus \mathbb{T}^3 there are 3 different slices describing 3 different mappings $\pi_1[U(1)] = \mathbb{Z}$ for each slice, such that the system is characterized by 3 different components of vector \vec{I} .

The definition (2) is very similar in structure to the conventional topological winding number in two-dimensional gauge field theories, such as the 2d Schwinger model, with nontrivial mapping $\pi_1[U(1)] = \mathbb{Z}$. It is assumed that the periodic boundary conditions are imposed on the angular variable α in definition (2), and therefore the topological invariant \vec{I} is conserved.

Few comments are in order. First, for the winding number \vec{I} to change, an entire path across the periodic cell must change. In non relativistic superfluid it could only happen as a result of tunnelling transitions (which is negligibly small effect for macroscopically large systems) or interactions of the system with fluctuating vortices. The reason why vortices may change the winding number is that they are characterized (locally) by vanishing superfluid density $n_s \simeq 0$ inside the vortex core, where the winding number can locally “unwind” itself. Such an interaction can in principle change the topological invariant (2) by transferring it to the vortices. Therefore, our comment is that the coupling of the winding state classified by $\vec{I} \neq 0$ with the vortices may, in principle, transfer the winding number from the bulk to a boundary. It could only happen when a sufficiently large number of coherent vortices represented by a macroscopically large proliferated vortex loop is present in the system [3]. It

is quite obvious that at zero, or very low temperature, when there are very few vortices present in the system the topological invariant (2) is conserved as the superfluidity is protected by the topological arguments. Only at sufficiently large temperature at $T \simeq T_c$ a percolated vortex network may emerge and remove the winding number to the boundary. It is exactly the temperature when the phase transition occurs.

This picture when the winding number \vec{I} is conserved should be contrasted with relativistic quantum field theories, such as QCD, where the tunnelling transitions constantly and continuously occur all the time, selecting a specific $|\theta\rangle$ vacuum state, in contrast with non relativistic systems where the ground state is normally classified by the conserved winding number $|l\rangle$, rather than by θ parameter representing a superposition of different winding states.

In fact, as emphasized in [3] the phenomenon of superfluidity itself is almost a trivial consequence of topological features of a complex scalar field defined on a nontrivial manifold \mathbb{M} with at least one nontrivial mapping $\pi_1[U(1)] = \mathbb{Z}$. This scalar field satisfies the Gross-Pitaevskii equations and effectively describes a superfluid liquid, as will be reviewed in next section III.

To reiterate: the superfluidity itself is a relatively simple problem of *classical* field theory of a scalar complex field governed by Gross-Pitaevskii Lagrangian. A hard problem in this system is understanding of the mechanisms of how the superfluidity is getting destroyed by some quantum or thermal fluctuations. The relevant dynamics must include, in one way or another, the fluctuations which carry the topological vorticity and which are capable to remove or destroy the winding number, as explained above. Precisely this problem on possible mechanisms of the phase transition between a normal $\vec{I} = 0$ and superfluid $\vec{I} \neq 0$ states is the main subject of the present work.

III. SUPERFLUIDITY: PHONONS AND ROTONS.

We now want to review the well known portion of the effective Lagrangian describing the Goldstone modes related to spontaneous violation of the global $U(1)$ symmetry. The starting point is Gross-Pitaevskii description when the superfluidity is described in terms of a single scalar complex field Φ with non-vanishing expectation value $\langle \Phi \rangle = n_S$, while its phase describes the corresponding Goldstone boson

$$L_{GP} = -\frac{i}{2} (\Phi^* \partial_t \Phi - \Phi \partial_t \Phi^*) - \frac{1}{2m} |\vec{\nabla} \Phi|^2 - \frac{\lambda}{2} (|\Phi|^2 - n_S)^2. \quad (3)$$

We define the Goldstone boson as the phase of the scalar field $\Phi = \sqrt{n} \exp(i\alpha)$ such that the relevant part of the

Lagrangian assumes the form

$$L_{GP} = (n\partial_t\alpha) - \frac{n}{2m}(\vec{\nabla}\alpha)^2 - \frac{\lambda}{2}(n - n_S)^2. \quad (4)$$

Integrating out the heavy $n(t, \vec{r})$ field, which corresponds to substitution

$$n \simeq n_S + \frac{1}{\lambda} \left[\partial_t\alpha - \frac{1}{2m}(\vec{\nabla}\alpha)^2 \right], \quad (5)$$

leads to the following conventional Lagrangian describing the massless Goldstone field $\alpha(t, \vec{r})$

$$L_G = (n_S\partial_t\alpha) + \frac{1}{2\lambda}(\partial_t\alpha)^2 - \frac{n_S}{2m}(\vec{\nabla}\alpha)^2 + \text{interactions}. \quad (6)$$

The quadratic terms in eq. (6) describe the massless Goldstone field $\alpha(t, \vec{r})$ with dispersion relation $\omega \sim k$. The velocity field $\vec{v}(t, \vec{r})$ introduced earlier is related to the Goldstone field as follows, $\vec{g}(t, \vec{r}) = m\vec{v} = \vec{\nabla}\alpha$. As a result of this relation one can check that the behaviour of the winding number I_i can be expressed in terms of the velocity field as equation (2) states.

A generalization of this analysis when the entire background is slowly moving with nonzero velocity $\vec{v}_{\text{slow}} \neq 0$ or experience a rotation with $\Omega \neq 0$ is straightforward. The resulting effective Lagrangian can be written in terms for the massless Goldstone fields propagating in the curved background

$$L_G = \frac{1}{2} \sqrt{-g} g^{\mu\nu} \partial_\mu \alpha \partial_\nu \alpha + \text{interactions}, \quad (7)$$

where $g^{\mu\nu}$ is the so-called induced acoustic metric which can be explicitly computed in terms of the original parameters of the theory [4].

The first term in (6) is a total derivative and does not change the equation of motion for the Goldstone $\alpha(t, \vec{r})$ field. However, this term can not be ignored as it will rise to a topological phase. In fact, in all respects this term is similar to the Berry phase. Furthermore, one can show that this term can be interpreted as a source for the Magnus forces for a moving vortex [4, 5]. Intuitively, this is expected result as first term in (6) is similar, in all respects, to the phase factor that would be generated by a charged particle moving in a uniform magnetic field with the action $e \oint dt \dot{x}_i A_i$.

The phonons reviewed above are not the only quasi-particles present in the system. Another type of quasi-particles, are the so-called rotons [6, 7]. The roton's properties can be studied by analyzing the so-called form-factor $S(\vec{k})$ defined as follows

$$S(\vec{k}) = \int d^3r \delta n(\vec{r}) e^{i\vec{k} \cdot \vec{r}}, \quad \delta n \simeq (n - \langle n \rangle) \quad (8)$$

where $\delta n(\vec{r})$ describes the density fluctuations of atoms in the liquid, which in low energy effective description can be approximated by the effective field Φ according to eq. (3). In terms of form factor $S(\vec{k})$ the spectral properties of excitations can be expressed as follows

$$\epsilon(\vec{k}) = \frac{\hbar^2 \vec{k}^2}{2mS(\vec{k})}. \quad (9)$$

The form factor $S(\vec{k})$ is known experimentally from neutron scattering. It has the following features. For sufficiently small $|\vec{k}|$ the form factor $S(\vec{k})$ shows a linear scaling, $S(\vec{k}) \sim |\vec{k}|$. It can be identified with excitations of phonons (6), see original papers [6, 7] and the textbook [3] with nice historical comments. For linear $S(\vec{k}) \sim |\vec{k}|$ the dispersion relation (9) indeed exhibits the linear scaling $\epsilon(\vec{k}) \sim |\vec{k}|$ consistent with phonon's interpretation. Another profound feature of the form factor $S(\vec{k})$ is that it has a maximum at wave number $|\vec{k}_0| \simeq 2 \text{ \AA}^{-1}$. In vicinity of this maximum one can expand $S(\vec{k}) \simeq S_0(k_0) - \frac{1}{2}|S_0''|(\vec{k} - \vec{k}_0)^2$ such that $\epsilon(\vec{k})$ exhibits the behaviour corresponding to the gapped modes with dispersion relation

$$\epsilon(\vec{k}) \simeq \epsilon_0 + \frac{\hbar^2 \vec{k}_0^2 |S_0''| (\vec{k} - \vec{k}_0)^2}{4mS_0^2}, \quad \epsilon_0 \simeq \frac{\hbar^2 \vec{k}_0^2}{2mS_0}. \quad (10)$$

The corresponding gapped excitations have been identified with rotons [3, 6, 7]. It is important to emphasize that the gap ϵ_0 related to the rotons is a temperature dependent parameter, i.e. $\epsilon_0(T)$. However, while $\epsilon_0(T)$ slowly varies with the temperature, it does not vanish at the critical temperature $\epsilon_0(T_c) \neq 0$ and remains approximately constant $\epsilon_0(T \gg T_c) \approx 5K$ for the temperatures well above the critical T_c , see e.g. [8] for references on numerous experimental results. This property will play an important role in our following discussions, where we argue that another dynamical gap parameter emerges in the system, which however vanishes at $T = T_c$, and therefore cannot be identified with conventional roton's gap $\epsilon_0(T)$.

The rotons play a key role in formulation of the Landau criterion

$$\omega_k + \vec{v} \cdot \vec{k} > 0, \quad (\text{Landau criterion}) \quad (11)$$

which is commonly interpreted in the literature as a criterion for superfluidity. Furthermore, there is a wide spread opinion that the rotons characterized by typical dispersion relation (10) exist only in superfluid states.

There is a number of arguments why this interpretation cannot be correct. First of all, there is a numerous experiments which convincingly show that the phonon-roton spectrum is not unique for superfluidity, but in fact, is very generic characteristic of a liquid state. Examples include, but not limited to such systems as liquid titanium, normal (not superfluid) helium, molecular para-hydrogen, neon, oxygen in supercritical region, and many others, see e.g. [8] for references on the experimental results. Furthermore, it is well-known fact that the critical velocity calculations based on the measured roton's minimum is much higher than observed values by orders of magnitude.

An independent argument [3] which basically leads to the same conclusion is based on topological reasoning suggesting that the topological invariant (2) remains intact as long as large vortex loop is not generated in the system. The Landau criterion (11) obviously does not

carry any information about the large vortex loops in the system as it is formulated in terms of the local microscopical degrees of freedom: the rotons and phonons. Therefore, the topological arguments [3] also suggest that eq.(11) cannot serve as a criterion for superfluidity.

IV. NOVEL CRITERION FOR SUPERFLUIDITY

In our studies in the present work we shall use a different criterion for the superfluidity based on the correlation function for the winding numbers $\langle I^2 \rangle$ defined in terms of the topological invariant (2) with non-contractable Γ . The key observation of refs. [1–3] is that the corresponding correlation function is directly related to the superfluid density n_s , and therefore, can serve as a criterion for superfluidity. In what follows in all our constructions we always assume that we are dealing with non-contractable Γ when the topological invariant (2) is conserved.

The conservation of \vec{I} implies that one can introduce an intensive variable \vec{s} which is thermodynamically conjugated to \vec{I} such that the grand canonical potential can be represented as follows [3]:

$$\Omega(T, \mu, \vec{v}_n, \vec{s}) = -T \ln \mathcal{Z}, \quad \langle \vec{I} \rangle = -\frac{\partial \Omega}{\partial \vec{s}}. \quad (12)$$

The physical meaning of \vec{s} can be inferred from the following generic relation [3]:

$$\vec{s} = n_s(\vec{v}_s - \vec{v}_n). \quad (13)$$

From this relation it follows [3] that in the reference frame of the walls where $\vec{v}_n = 0$ one can express the superfluid density n_s as the static response of the system with respect to small variation of \vec{s} :

$$\frac{1}{n_s} = -\frac{1}{3mV} \nabla_{s_i}^2 \Omega = \frac{\langle \vec{I}, \vec{I} \rangle}{3mVT}, \quad \vec{s} \rightarrow 0, \quad (14)$$

where the correlation function $\langle \vec{I}, \vec{I} \rangle$ is defined as

$$\langle \vec{I}, \vec{I} \rangle = \lim_{\mathbf{k} \rightarrow 0} \int_{\mathbb{M}} d^3x \int_{\mathbb{M}} d^3x' e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')} d^3x' \cdot \langle \vec{\nabla} \alpha(\mathbf{x}), \vec{\nabla} \alpha(\mathbf{x}') \rangle. \quad (15)$$

where factor $\lim_{\mathbf{k} \rightarrow 0}$ must be included into the definition to keep only connected portion of the correlation function, see few comments at the end of this section. As we discuss below, a proper evaluation of the contact term of the correlation function (15) at $\Delta \mathbf{x} \equiv (\mathbf{x} - \mathbf{x}') \rightarrow 0$ requires that both \mathbf{x} and \mathbf{x}' lie on the same path $\Gamma \in \mathbb{M}$ when $\Delta \mathbf{x}$ approaches zero.

The expectation value in this expression should be computed using grand canonical potential defined by (12). Similar relation is known to exist in QCD where the vacuum energy E_{vac} is expressed in terms of the topological susceptibility $(\partial^2 E_{\text{vac}} / \partial \theta^2) \sim \langle Q^2 \rangle$ where Q is the topological density operator, and parameter θ is conjugated to Q enters the QCD partition function as θQ .

This QCD relation is the direct analog of eq. (14) when the vacuum energy E_{vac} in QCD plays the role of Ω , the topological susceptibility $\langle Q^2 \rangle$ is analogous to the winding correlation function (15), while the QCD parameter θ is analogous to intensive variable \vec{s} . We want to see if this analogy is sufficiently deep, and whether the correlation function $\langle I^2 \rangle$ in superfluid state has similar features which are known to be present in the topological susceptibility in strongly coupled QCD.

In what follows we want to be more specific and consider a simple manifold $\mathbb{M} = \mathbb{S}^1 \times \mathbb{I}^2$ which represents a hollow cylinder length L_z , internal radius R_1 and external radius R_2 . In other words, for simplicity we consider a manifold which has a single nontrivial mapping $\pi_1[U(1)] = \mathbb{Z}$ corresponding to one type of non-contractable contour Γ along \mathbb{S}^1 . For this topology the winding number of the system has a single nontrivial component and can be represented as follows

$$I_z = L_z(R_2 - R_1) \int_{\mathbb{S}^1} \partial_i \alpha dl_i = 2\pi n_z^{\text{class}} L_z(R_2 - R_1), \quad (16)$$

where $n_z^{\text{class}} \in \mathbb{Z}$ has the physical meaning of strength of vorticity (number of times the phase field $\alpha(x) \in U(1)$ makes a full circle after traversing a closed path along $\mathbb{S}^1 \in \mathbb{M}$). Factor $2\pi L_z(R_2 - R_1)$ represents the surface area of a hollow cylinder and can be interpreted as a corresponding degeneracy factor. Finally, we use subscript n^{class} to emphasize that this integer number classifies exclusively the classical portion of the winding number.

The winding (16) is a conserved quantity as we already explained. However, for the purposes of the present work we want to study the fluctuations of the winding number $\delta \vec{I}$ rather than its classical conserved portion (16). This is because the fluctuations of the vorticity may potentially change the winding number (16), which is precisely the main goal of our studies. Exactly these fluctuations of the vorticity in form of generation of a large vortex loop will eventually lead to the phase transition at T_c .

Formal manifestation of the quantum nature of the correlation function (15) which is the main focus of our studies, is expressed in terms of a factor $\exp(ikx)$ under the integral. The corresponding $\lim_{k \rightarrow 0}$ should be evaluated at the end of the computations. This formal procedure selects the connected portion of this correlation function entering expression (14).

V. THE TOPOLOGICAL VORTICES

As we explained in previous section the main subject of our studies is the physics of fluctuating vortices. We want to avoid confusion with notations in what follows and define a different winding number W_i which is the operator describing the quantum/thermal fluctuations of the vorticity, in contrast with classical expression (16) which represents a conserved topological number of the system. These two objects, of course, are related to each other with a simple numerical factor representing the size

of the system. However, we prefer to use a different normalization for the vortices in our discussions in this section to avoid some numerical dimensional factors such as size of the system entering (16). For thermally excited vortices to be studied below this size is essentially a microscopical size of the vortex's core rather than macroscopical size of a system. The vortices become very large in size, of the order of the system, when T approaches the critical value T_c .

Our definition for the winding number W_z let us say, in z direction assumes that we measure the vorticity by computing the contour integral along path Γ which lies on the xy plane such that the computations are reduced to evaluation of the surface integral $\partial S = \Gamma$. In particular, for a single vortex pointing along z direction it is given by

$$W_z = \int_{\Gamma} \partial_i \alpha(\vec{x}) \frac{dl_i}{2\pi} = \int_S \frac{dxdy}{2\pi} (\partial_1 \partial_2 - \partial_2 \partial_1) \alpha(\vec{x}) = n_z \int_S dxdy \delta_z^2(x_{\perp}). \quad (17)$$

The generalization of this topological description (17) for arbitrary vortex shape is straightforward: the winding number operator W_k can be expressed in terms of the circulation field $\gamma_k(\vec{x})$ as follows,

$$W_k = \frac{1}{2\pi} \int dS_k \gamma_k(\vec{x}), \quad (18)$$

where $\gamma_k(\vec{x})$ is defined as

$$\gamma_k(\vec{x}) = \epsilon_{ijk} \partial_i g_j(\vec{x}) = \epsilon_{ijk} \partial_i \partial_j \alpha(\vec{x}), \quad (19)$$

where $g_j(\vec{x}) = \partial_j \alpha(\vec{x}) = mv_j(\vec{x})$ is the velocity field introduced earlier. In particular, for an ideal structureless singular vortex (17) one has

$$\gamma_k(\vec{x}) = \epsilon_{ijk} \partial_i g_j(\vec{x}) = 2\pi \delta_k^2(x_{\perp}). \quad (20)$$

The winding W_k counts number of crossing points between the vortices and the surface S_k . The physical meaning of $\gamma_k(\vec{x})$ is quite obvious– it describes the density of circulation per unit area along the vortex of arbitrary geometrical shape pointing locally in k direction. This function obviously satisfies the conservation law, $\partial_k \gamma_k(\vec{x}) = 0$ for closed vortex lines even for the singular ones. This is a formal expression of the property that the winding number (17), (18) does not depend on position of the cut along the vortex. If a vortex line is open with the ends at points \vec{x}_i and \vec{x}_f than one can easy to see that $\partial_k \gamma_k(\vec{x}) = 2\pi [\delta^3(\vec{x} - \vec{x}_i) - \delta^3(\vec{x} - \vec{x}_f)]$. For closed vortices one has $\vec{x}_i = \vec{x}_f$ and one returns to the conservation law, $\partial_k \gamma_k(\vec{x}) = 0$.

One should emphasize that our fluctuating vortices are not straight lines all pointing in the same direction, which is a conventional picture for rotating fluid. Rather, the picture we have in mind is that the fluctuating vortices look like a complicated dynamical mixture of knotted, crumpled and wrinkled fluctuated spaghetti, with the only difference is that these spaghetti make closed, rather

then opened vortices, and they also have specific chiralities.

When we cut our spaghetti-like system, let us say, in z direction, we observe a large number of plus and minus fluxes sitting on our (xy) - slice where we made a cut. A similar picture would emerge if we cut in x or y directions. This is precisely the reason why eq. (2) exhibits three different conserved winding numbers I_i , and corresponding conjugated variable \vec{s} which enters the definition (12) is a 3-vector, rather than a scalar. It should be contrasted with non-abelian gauge QCD when there is a single topological classification scheme based on $\pi_3[SU(N)] = \mathbb{Z}$ and there is a single conjugated parameter θ which plays the same role as intensive parameter \vec{s} plays in superfluid systems.

It is very instructive to present an analogy with magnetism as it gives a good intuitive picture about superfluid vortices and their description in terms of our Goldstone field $\alpha(\vec{x})$, velocity field $g_j(\vec{x}) = \partial_j \alpha(\vec{x})$ and circulation density field $\gamma_k(\vec{x}) = \epsilon_{ijk} \partial_i g_j(\vec{x})$. To be more precise, our field $g_i(\vec{x})$ behaves in all respects as the magnetic field $B_i(\vec{x})$. Indeed the magnetic field satisfies the following equations

$$\partial_i B_i = 0, \quad \epsilon_{ijk} \partial_i B_j(\vec{x}) = \mu_0 J_k(\vec{x}), \quad \partial_k J_k(\vec{x}) = 0. \quad (21)$$

Equation $\partial_i B_i = 0$ is similar to $\partial_i g_i = m \vec{\nabla} \cdot \vec{v}_s = 0$ which represents the incompressibility of superflow. At the same time equation $\epsilon_{ijk} \partial_i B_j(\vec{x}) = \mu_0 J_k$ is similar to (19), (20) where the role of $\mu_0 J_k$ plays $\gamma_k(\vec{x})$. The conservation of the current in magnetism $\partial_k J_k(\vec{x}) = 0$ is expressed in terms of the property $\partial_k \gamma_k(\vec{x}) = 0$ in our spaghetti-like system. The summary of this analogy can be represented in one line as follows

$$\partial_i g_i = 0, \quad \epsilon_{ijk} \partial_i g_j(\vec{x}) = \gamma_k(\vec{x}), \quad \partial_k \gamma_k(\vec{x}) = 0, \quad (22)$$

which is formally coincides with (21). The last equation represents the invariance of the winding number of the system W_k with respect to position of the cut of the spaghetti environment which defines a specific slice perpendicular to the k - direction. The circulation field $\gamma_k(\vec{x})$ in spaghetti system plays the role of the current distribution $J_k(\vec{x})$ in a knotted and twisted system of wires in the magnetism.

One can make one more step in this formal analogy and introduce new vector potential (in fact the axial potential) $a_j(\vec{x})$ such that

$$\epsilon_{ijk} \partial_i a_j(\vec{x}) \equiv g_k(\vec{x}), \quad (23)$$

similar to $\vec{B} = \vec{\nabla} \times \vec{A}$. The field $a_j(\vec{x})$ obviously is not uniquely defined as the corresponding “gauge transformation” $a_j(\vec{x}) \rightarrow a_j(\vec{x}) - \partial_j \Lambda(\vec{x})$ leave the physical velocity field $g_k(\vec{x})$ unaltered. Using the “Coulomb” gauge $\partial_k a_k = 0$ one can compute the velocity field $g_k(\vec{x})$ in terms of the classical distribution of circulation sources $\gamma_k(\vec{x})$ as it is normally done in magnetism. To be more precise, the potential $a_j(\vec{x})$ according to eqs. (19), (23) satisfies the following Poisson equation

$$\vec{\nabla}^2 \vec{a} = -\vec{\gamma}, \quad (24)$$

which has the following solution in terms of known Green's function

$$a_k(\vec{x}) = \frac{1}{4\pi} \int d^3x' \frac{\gamma_k(\vec{x}')}{|\vec{x} - \vec{x}'|}. \quad (25)$$

In case of infinitely thin single ideal vortex loop γ_L with unit quanta (1) of circulation formula (25) assumes the form

$$a_k(\vec{x}) = \frac{1}{2} \oint_{\gamma_L} dx'_k \frac{1}{|\vec{x} - \vec{x}'|}, \quad (26)$$

where we used relation (20) for $\gamma_k(\vec{x})$. In formula (26) the contour integral is taken along the vortex line γ_L .

The vector (axial) potential (26) is long-ranged, similar to conventional vector potential \vec{A} in magnetism. This long-ranged interaction leads to the vortex-vortex interaction similar to Biot-Savart law for the magnetism. To be more precise, for continuous distribution of circulation $\gamma_k(\vec{x})$ the interaction energy between vortices can be represented in the following form

$$E_{\text{int}} = \frac{\sigma}{2} \int \frac{d^3x d^3x'}{4\pi} \frac{\gamma_k(\vec{x}) \gamma_k(\vec{x}')}{|\vec{x} - \vec{x}'|}, \quad (27)$$

where $\sigma = n_s/m$ for an ideal infinitely long and infinitely thin vortex. In what follows we want to treat σ as a free parameter which determines the strength of vortex-vortex interaction with a finite size of the core. It is convenient for this purpose to introduce a dimensionless parameter $\eta \sim 1$ as follows $\sigma = \eta n_s/m$ such that all deviations from the ideal case is coded by parameter η . It obviously depends on an internal vortex structure as well as on its relation to a typical inter-vortex distance at temperature T .

Using expression (25) for the vector potential $a_k(\vec{x})$ the formula for E_{int} can be also written in the local form

$$E_{\text{int}} = \frac{\sigma}{2} \int d^3x a_k(\vec{x}) \gamma_k(\vec{x}). \quad (28)$$

Formula (27) obviously implies that two parallel infinitely long vortices repel each other, in contrast with magnetism when the parallel wires with the currents in the same direction attract each other, though the formal expression for magnetic case is identically the same as (27). The difference is of course due to the law of induction when the electric field is induced under a small variation of the system. Our superfluid vortices are not subject to the law of induction, and therefore they repel.

Once the vector potential $a_k(\vec{x})$ is known, the corresponding physical velocity field $g_k(\vec{x})$ can be computed from (23) as usual. Therefore, for any distribution of vortices the corresponding classical velocity field can be explicitly computed. However, our vortices are complicated objects, and simplified description of infinitely long straight vortices is not quite appropriate description for questions we want to address. Nevertheless, the algebraic and topological structure of the fields (22), (23) we introduced in this section will play an important role in the analysis which follows.

The correlation function $\langle I^2 \rangle$ which enters the fundamental relation (14) is proportional to the following correlation function (up to some normalization factors)

$$\langle W_k, W_k \rangle = \lim_{k \rightarrow 0} \int e^{i\vec{k} \cdot (\vec{x} - \vec{x}')} \frac{dS_k}{2\pi} \frac{dS'_k}{2\pi} \langle \gamma_k(\vec{x}), \gamma_k(\vec{x}') \rangle \quad (29)$$

The corresponding expectation value $\langle W^2 \rangle$ does not depend on the position of the slice dS_k along k direction as a result of exact property $\partial_k \gamma_k(\vec{x}) = 0$ as explained above. It implies that $\langle I^2 \rangle$ also holds this feature.

VI. SUPERFLUIDITY AND THE WINDING NUMBER SUSCEPTIBILITY $\langle I^2 \rangle$.

The main goal of the present section is to derive and analyze the effective action which accounts for the complicated dynamics of the vortices introduced above. Using the corresponding action we compute the expectation value of the winding susceptibility $\langle I^2 \rangle$, which is the main object of our studies. A hope is that these computations will provide us with a hint on nature of configurations which destroy the superfluidity. Therefore, the corresponding vortex configurations which are responsible for destroying the superfluidity can be thought as the configurations which control the phase transition at T_c .

The technique which will be used in the present studies is well established one, and it is commonly known to particles physics and condensed matter communities. For convenience of the readers we review this technique in Appendix A which was previously developed for a QCD-like model with the main purpose to emphasize on some amazing similarities between the two systems. In particular, we show how the mass gap is generated in both systems as a result of mixture of the Goldstone mode with an auxiliary topological field.

We should warn a reader from the start that we have made a large number of assumptions and approximations in course of our analysis in this section. In particular, it is quite obvious that the dynamics of the vortices, accounting for their numerous configurations with different densities and topologies is very important for computations of $\langle I^2 \rangle$. Such a computation is obviously a very ambitious goal which is well beyond the scope of the present work. We shall specifically formulate those assumptions along the course of our presentation. However, we believe that the basic consequences of this framework are quite generic and robust and not very sensitive to those approximations.

The organization of this section is as follows. In subsection VIA we introduce the auxiliary non-propagating topological field and derive the corresponding effective action, similar to the Chern Simons term which is a conventional technique in studies of the topologically ordered phases.

Using this topological action we compute in subsection VIB the winding susceptibility $\langle I^2 \rangle$ and analyze its properties. The main feature of this correlation function

is a generation of the mass gap $\Delta_\eta(T)$ as shown in Section VIC. The obtained expression for the gap is highly sensitive to superfluid density: it vanishes at the critical temperature $T = T_c$ when $n_S = 0$. This unique feature obviously implies that this gap cannot be identified with the roton's excitations which are present in the system above and below the critical temperature. Therefore, we dubbed the corresponding excitations as vortons, which is the subject of Section VII.

One should comment here that the obtained behaviour of $\langle I^2 \rangle$ is very similar to known properties of $\langle Q^2 \rangle$ which plays a crucial role in resolution of the so-called $U(1)_A$ problem in QCD. In both cases the mass is generated as a result of mixing of the Goldstone mode with a non-propagating topological auxiliary field. In weakly coupled “deformed QCD” all claims can be made very precise as reviewed in Appendix A for convenience.

A. Superfluidity and auxiliary gauge fields

The first step to proceed with this program is to construct the current density $j_k(\mathbf{x})$ for a generic configuration of the $n^{(i)}$ vortices (20) directed along k -th component at point $\mathbf{x}_\perp^{(i)}$ as expressed below

$$j_k(\mathbf{x}) = \sum_i 2\pi n^{(i)} \delta_k^2(\mathbf{x}_\perp - \mathbf{x}_\perp^{(i)}(x_k)), \quad (30)$$

where the position of the i -th vortex $\mathbf{x}_\perp^{(i)}(x_k)$ obviously depends on x_k coordinate. In this formula we obviously ignored the internal structure of the vortices by assuming that the vortices are infinitely thin.

This current is analogous in all respects to the topological density operator in QCD describing a generic configuration of the point-like monopoles with arbitrary positions and colour orientations, see Appendix A for the details.

Our next step is to insert the delta function into the path integral with the field $b_i(\mathbf{x})$ acting as a Lagrange multiplier

$$\delta\left(j_k(\mathbf{x}) - \frac{[\epsilon_{ijk}\partial_i g_j(\mathbf{x})]}{2\pi}\right) \sim \int \mathcal{D}[b_i] e^{i \int d^3x b_k(\mathbf{x}) \cdot \left(j_k(\mathbf{x}) - \frac{1}{2\pi} [\epsilon_{ijk}\partial_i g_j(\mathbf{x})]\right)}, \quad (31)$$

where $j_k(\mathbf{x})$ in this formula is treated as the original expression (30) for the density circulation expression including the fast velocity field, while $\epsilon_{ijk}\partial_i g_j(\mathbf{x})$ is treated as a slow-varying external source describing the long distance physics, similar to the treatment in Appendix A of the auxiliary non-propagating topological fields in QCD. Furthermore, our formula (31) corresponds to the normalization when the current $j_k(\mathbf{x})$ carries integer fluxes of circulation.

Our task now is to integrate out the original fluctuating vortices with different shapes (treated here as the fast degrees of freedom) and describe the large distance

physics in terms of effective slow varying fields represented by the Lagrange multiplier $b_i(\mathbf{x})$ and the auxiliary $g_i(\mathbf{x})$ field. One should emphasize that $g_i(\mathbf{x})$ field in formula (31) should not be confused with local velocity field $\vec{g}(\mathbf{x}) = m\vec{v}$ in our previous discussions. The difference is that $(b_k(\mathbf{x}), g_i(\mathbf{x}))$ as well as $\gamma_k(\mathbf{x}) = [\epsilon_{ijk}\partial_i g_j(\mathbf{x})]$ fields entering formula (31) will effectively describe the long range behaviour of the system after the integrating out the original fast fluctuating vortices with different shapes and positions, while the original $g_i(\mathbf{x})$ field describes a specific fast fluctuating local velocity distribution for a given vortex configuration. We keep the same notations because in what follows we will be dealing exclusively with long-ranged effective $[b_k(\mathbf{x}), g_i(\mathbf{x})]$ auxiliary fields.

To construct the partition function \mathcal{Z} one should integrate out the original large fluctuating vortices with different shapes, which is obviously a very hard technical problem. This task looks even harder because we should do these computations in the background of long range $[b_k(\mathbf{x}), g_i(\mathbf{x})]$ auxiliary fields. A similar, but technically less complicated problem (when one should integrate over all positions and colour orientations of the monopoles) has been explicitly carried out in the so-called “deformed QCD” model, see Appendix A for the details and comparison with present analysis. One part of the effective action $S_{\text{top}}[b_k(\mathbf{x}), g_i(\mathbf{x})]$ though can be established immediately from (31) as it represents the topological portion of the action and it is expressed exclusively in terms of auxiliary long ranged auxiliary fields,

$$\mathcal{Z} \sim \int \mathcal{D}[b] \mathcal{D}[g] \mathcal{D}[\alpha] e^{-\beta S[b, g, \alpha] - S_{\text{top}}[b, g]} \quad (32)$$

$$S_{\text{top}}[b, g] = \frac{i}{2\pi} \int_{\mathbb{M}} d^3x b_k(\mathbf{x}) [\epsilon_{ijk}\partial_i g_j(\mathbf{x})].$$

The computations of $S[b, g, \alpha]$ is much more complicated task as we already mentioned. We should remark here that α field is included into the effective action $S[b, g, \alpha]$ because it describes the long range light Goldstone field (6) which must remain in the action after integrating out all heavy degrees of freedom and complicated net of vortices. As explained above, in a class of simple models such kind of computations can be explicitly carried out and compared with independent exact results as discussed in Appendix A. Our present case obviously does not belong to this class (of simple models) where exact computations are feasible. However we can establish the structure of $S[b, g, \alpha]$ from symmetry reasoning.

The argument goes as follows. The current (30) which describes density of circulation as discussed after equations (19), (20) is exactly conserved for any closed vortices. This conservation is manifestation of formal property of the system that the winding number remains the same for any slice along any direction. The corresponding current expressed in terms of the auxiliary long range field is also conserved, $\partial_k [\epsilon_{ijk}\partial_i g_j(\mathbf{x})] = 0$. Therefore, the Lagrange multiplier $b_k(\mathbf{x})$ field is in fact a “gauge” auxiliary field such that the “gauge transformation”

$$b_k(\mathbf{x}) \rightarrow b_k(\mathbf{x}) - \partial_k \Lambda(\mathbf{x}) \quad (33)$$

leave the partition function (31) due to large vortices unaltered. This “gauge invariance” unambiguously fixes a possible structure of the long distance action $S[b, g, \alpha]$ entering (32). Indeed, the auxiliary “gauge” field may enter the Gross-Pitaevskii Lagrangian (3) only in combination

$$\nabla_i \Phi(\mathbf{x}) \rightarrow [\nabla_i - ib_i(\mathbf{x})] \Phi(\mathbf{x}). \quad (34)$$

After integrating out all the heavy degrees of freedom this “gauge shift” implies that the relevant Hamiltonian describing the Goldstone field (6) assumes the following form

$$H_G = \frac{n_S}{2m} \int_{\mathbb{M}} d^3x [\partial_i \alpha(\mathbf{x}) - b_i(\mathbf{x})]^2 + \text{interactions}, \quad (35)$$

where we neglected higher order corrections in fields and derivatives which follow from the expansion of the Gross-Pitaevskii Lagrangian (3).

While the local interaction terms can be neglected for the qualitative analysis, the long-ranged term which is similar to the instantaneous coupling in electrodynamics which has the form (27) must be included into the consideration. The simplest way to account for this long-ranged interaction is to represent it in terms of the auxiliary vector potential $a_k(\mathbf{x})$ which is related to the auxiliary $g_j(\mathbf{x})$ field precisely in the same way as original fast fluctuating fields as given by eq.(23). Therefore, the interaction term (27) can be written as follows

$$H_{\text{int}} = -\frac{\sigma}{2} \int_{\mathbb{M}} d^3x a_k(\mathbf{x}) \vec{\nabla}^2 a_k(\mathbf{x}). \quad (36)$$

Similarly, the topological term (32) can be also rewritten in terms of the “gauge” field $a_k(\mathbf{x})$ as follows

$$S_{\text{top}}[b, a] = -\frac{i}{2\pi} \int_{\mathbb{M}} d^3x b_k(\mathbf{x}) \vec{\nabla}^2 a_k(\mathbf{x}), \quad (37)$$

where we always assume the “Coulomb gauge” $\partial_k a_k(\mathbf{x}) = 0$ for the auxiliary $a_k(\mathbf{x})$ gauge field.

We are now in position to collect all relevant terms and represent the partition function to be used in following sections as follows

$$\begin{aligned} \mathcal{Z} &\sim \int \mathcal{D}[b] \mathcal{D}[a] \mathcal{D}[\alpha] e^{-\beta(H_G[b, \alpha] + H_{\text{int}}[a] + H_{\text{top}}[b, a])} \\ \beta H_{\text{top}}[b, a] &= -\frac{i}{2\pi} \int_{\mathbb{M}} d^3x b_k(\mathbf{x}) \vec{\nabla}^2 a_k(\mathbf{x}), \\ H_{\text{int}}[a] &= -\frac{\sigma}{2} \int_{\mathbb{M}} d^3x a_k(\mathbf{x}) \vec{\nabla}^2 a_k(\mathbf{x}), \\ H_G[b, \alpha] &= \frac{n_S}{2m} \int_{\mathbb{M}} d^3x [\partial_i \alpha(\mathbf{x}) - b_i(\mathbf{x})]^2. \end{aligned} \quad (38)$$

Few comments are in order. First of all, in formula (38) we neglected the dynamics of the vortices, including their mutual interactions, their numerous configurations with different densities, shapes and topologies, their internal structure, etc. We also ignored all types of interactions except for the crucial inter-vortex interaction term $\sim \sigma$. The accounting of the corresponding interaction effects is obviously a very ambitious goal which

is well beyond the scope of the present work. In principle, the corresponding computations can be carried out using the “vortex-ring model” [9], which basically employs the renormalization group technique, or explicit numerical simulations [10]. However, we believe that all these complications may change the numerical parameters which enter the expression for $\langle I^2 \rangle$ as discussed in Section VI C. However, they cannot drastically change the basic algebraic structure and the basic features of eq.(51) which represents the main focus of the present analysis.

Our next comment is as follows. We treated the Lagrange multiplier field $b_k(\mathbf{x})$ in eq.(38) as auxiliary, non-propagating gauge potential. In other words, we assume that the gauge invariant combination $(\partial_i b_j(\mathbf{x}) - \partial_j b_i(\mathbf{x}))$, which is allowed by the symmetry, nevertheless cannot be generated as a result of any interactions.

Our next remark is as follows: the topological action with imaginary i which enters (38) should not be considered as a signal of violation of unitarity. These fields do not have conventional kinetic terms, and do not propagate. The fields $[b_k(\mathbf{x}), a_k(\mathbf{x})]$ should be treated as an auxiliary fields saturating the path integral, similar to computation of a conventional integral using a steepest descent approximation when a saddle point lies in a complex plane outside the region of definition of the original variables. Analogous effect also occurs in “deformed QCD” model reviewed in Appendix A, where similar “imaginary” action (A12) reproduces the exact correlation functions obtained by a different independent approach.

Our final comment is about the “gauge invariance” expressed by eqs.(33),(34). Similar structure has been discussed previously in the literature [3], [11]. The difference is that the gauge vector potential in [3], [11] is essentially an external background field representing the twisted boundary conditions. In contrast, our gauge field $b_k(\mathbf{x})$ is introduced as a Lagrange multiplier, and it is the fluctuating (though non-propagating), field which describes the dynamics of internal fluctuating vortices with variety of shapes and positions. In our framework one and the same “gauge field” $b_k(\mathbf{x})$ enters not only the Goldstone portion of the Hamiltonian, $H_G[b, \alpha]$, but it also enters the topological portion of the Hamiltonian $H_{\text{top}}[b, a]$ according to eq.(38).

B. Winding number susceptibility $\langle I^2 \rangle$

The main goal of this subsection is to compute the winding number susceptibility $\langle I^2 \rangle$ defined by eq. (15) using the partition function derived in previous subsection and given by (38). We follow the textbook [3] and define the “gauge invariant” winding number susceptibil-

ity

$$\langle I_k, I_k \rangle = \lim_{\mathbf{k} \rightarrow 0} \int_{\mathbb{M}} d^3x \int_{\mathbb{M}} d^3x' e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')} \langle (\partial_k \alpha(\mathbf{x}) - b_k(\mathbf{x})), (\partial_k \alpha(\mathbf{x}') - b_k(\mathbf{x}')) \rangle. \quad (39)$$

which respects the symmetry (33). As explained above this “gauge invariance” is a consequence of the “current conservation” $\partial_k \gamma_k(\mathbf{x}) = 0$. The expectation value (39) should be computed using the partition function defined by (38). The corresponding calculations can be carried out exactly because the Hamiltonian is quadratic in our approximate treatment.

To proceed with computations, we first integrate out the auxiliary $b_k(\mathbf{x})$ field which enters (38) as a Lagrange multiplier. It gives the constraint-like relation:

$$[b_k(\mathbf{x}) - \partial_k \alpha(\mathbf{x})] = i \left(\frac{mT}{n_s} \right) \frac{\vec{\nabla}^2 a_k(\mathbf{x})}{2\pi}. \quad (40)$$

Now the susceptibility is expressed exclusively in terms of the topological $a_k(\mathbf{x})$ field:

$$\begin{aligned} \langle I_k, I_k \rangle &= \lim_{\mathbf{k} \rightarrow 0} \int_{\mathbb{M}} d^3x \int_{\mathbb{M}} d^3x' e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')} \mathcal{I}(a_k) \quad (41) \\ \mathcal{I}(a_k) &= - \left(\frac{mT}{n_s} \right)^2 \left\langle \frac{\vec{\nabla}^2 a_k(\mathbf{x})}{2\pi}, \frac{\vec{\nabla}^2 a_k(\mathbf{x}')}{2\pi} \right\rangle. \end{aligned}$$

Our next step is to eliminate the Lagrange multiplier $b_k(\mathbf{x})$ in the Hamiltonian (38) using constraint (40). The corresponding simple algebraic manipulations lead to the following quadratic form for the Hamiltonian $H_{\text{tot}}[a, \alpha]$

$$\begin{aligned} \frac{H_{\text{tot}}[a, \alpha]}{T} &= \left(\frac{mT}{2n_s} \right) \cdot \int_{\mathbb{M}} d^3x \left(\frac{\vec{\nabla}^2 a_k(\mathbf{x})}{2\pi} \right)^2 \quad (42) \\ &- \int_{\mathbb{M}} d^3x \left[\frac{\sigma}{2T} a_k(\mathbf{x}) \vec{\nabla}^2 a_k(\mathbf{x}) + \frac{i}{2\pi} \partial_k \alpha(\mathbf{x}) \vec{\nabla}^2 a_k(\mathbf{x}) \right]. \end{aligned}$$

To complete the computations of the correlation function (41) one should diagonalize the quadratic form (42) using conventional trick to shift the variables from $a_k(\mathbf{x})$ to $a'_k(\mathbf{x})$:

$$\left(\frac{\vec{\nabla}^2 a_k(\mathbf{x})}{2\pi} \right) = \left(\frac{\vec{\nabla}^2 a'_k(\mathbf{x})}{2\pi} \right) + i \left(\frac{n_s}{mT} \right) \partial_k \alpha(\mathbf{x}). \quad (43)$$

This change of variables brings the Hamiltonian (42) to the desired diagonal form

$$\begin{aligned} \frac{H_{\text{tot}}[a, \alpha]}{T} &= \left(\frac{mT}{2n_s} \right) \cdot \int_{\mathbb{M}} d^3x \left(\frac{\vec{\nabla}^2 a'_k(\mathbf{x})}{2\pi} \right)^2 \quad (44) \\ &+ \int_{\mathbb{M}} d^3x \left[\left(\frac{n_s}{2mT} \right) (\partial_k \alpha(\mathbf{x}))^2 - \frac{\sigma}{2T} a'_k(\mathbf{x}) \vec{\nabla}^2 a'_k(\mathbf{x}) \right]. \end{aligned}$$

Before we proceed with computations we want to make few technical comments. First, we neglected the singular term $\sim \partial_k \alpha(\mathbf{x}) \frac{1}{\vec{\nabla}^2} \partial_k \alpha(\mathbf{x})$ which formally emerges from vortex-vortex interaction $\sim \sigma$. Naively, it could be

interpreted as a generation of the mass for the Goldstone field $\sim \alpha^2$ if we formally cancel the singularity $\partial_k \partial_k \frac{1}{\vec{\nabla}^2} \sim 1$. Such a manipulation with singular operator formally leading to α^2 term is obviously incorrect. The singularity emerges as a result of our ignorance of the vortex structure at small distances. Simple way to account for this structure is to replace $\frac{1}{\vec{\nabla}^2} \rightarrow \frac{1}{\vec{\nabla}^2 - \mu^2}$ with some typical mass scale μ . This cutoff procedure obviously leads to a desired result that the Goldstone field enters the Hamiltonian only through the derivative $\sim \partial_k \alpha(\mathbf{x})$ without violating any fundamental theorems. In deriving (44) we also neglected the terms which can be integrated by parts. Finally, we dropped the terms which vanish as a result of the “Coulomb gauge” choice, $\sim \partial_k a_k(\mathbf{x}) = 0$.

Now we are in position to proceed with computations of the susceptibility $\langle I^2 \rangle$ defined by (39), (41). We express the original fields $a_k(\mathbf{x})$ in terms of the shifted variables (43) to arrive to

$$\begin{aligned} \langle I^2 \rangle &= \lim_{\mathbf{k} \rightarrow 0} \int_{\mathbb{M}} d^3x \int_{\mathbb{M}} d^3x' e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')} [\mathcal{I}(\alpha) + \mathcal{I}(a'_k)] \quad (45) \\ \mathcal{I}(\alpha) &= \langle \partial_k \alpha(\mathbf{x}), \partial_k \alpha(\mathbf{x}') \rangle \\ \mathcal{I}(a_k) &= - \left(\frac{mT}{n_s} \right)^2 \left\langle \frac{\vec{\nabla}^2 a'_k(\mathbf{x})}{2\pi}, \frac{\vec{\nabla}^2 a'_k(\mathbf{x}')}{2\pi} \right\rangle, \end{aligned}$$

where the corresponding expectation values must be computed using Hamiltonian (44). The computations are straightforward because the Hamiltonian (44) is quadratic. From now on we drop the prime-sign in $\mathcal{I}(a_k)$ to simplify notations.

It is very instructive to compute $\mathcal{I}(\alpha)$ separately by first ignoring the term $\mathcal{I}(a'_k)$ related to auxiliary gauge fields. The result is

$$\mathcal{I}(\alpha) = \frac{mT}{n_s} \cdot \left(\frac{\delta(x - x')}{L_2 L_3} + \frac{\delta(y - y')}{L_1 L_3} + \frac{\delta(z - z')}{L_2 L_1} \right), \quad (46)$$

where the first term is computed assuming the path $\Gamma \in \mathbb{M}$ goes along \hat{x} direction, while the second and third terms are generated due to the paths along \hat{y} and \hat{z} directions correspondingly as explained after eq. (15). Indeed, the correlation function $\mathcal{I}(\alpha)$ obviously vanishes at $\mathbf{x} \neq \mathbf{x}'$ as $\vec{\nabla}^2 \frac{1}{|\mathbf{x} - \mathbf{x}'|} = 0$ at $\mathbf{x} \neq \mathbf{x}'$ in infinite volume. To get a better idea about the structure of the singularity for finite manifold \mathbb{M} when a singular limit is approached along a specific \hat{x} direction it is convenient to approximate the relevant portion of the Hamiltonian (44) as $\int dx \left(\frac{n_s L_2 L_3}{2mT} \right) (\partial_x \alpha(x))^2$. This Hamiltonian precisely generates the first singular term in eq. (46). Two other terms can be obtained in a similar manner.

The result (46) is well anticipated contact term. The corresponding contribution to $\langle I^2 \rangle_\alpha$ after integration over volume $\int d^3x d^3x'$ is

$$\langle I^2 \rangle_\alpha = \int d^3x \int d^3x' \cdot \mathcal{I}(\alpha) = \left(\frac{3mT}{n_s} \right) V. \quad (47)$$

Few comments are in order. First of all, if we substitute (47) to our original formula (14) for n_s we get

identity $n_s = n_s$, which is expected result as we neglected many important elements, including $\mathcal{I}(a_k)$ term, and all the interaction terms. We also neglected the internal vortex structure, interaction of the Goldstone field with the vortices, etc, which in principle can be recovered from the original Gross-Pitaevskii Lagrangian (3). Nevertheless, this “trivial” result shows that our formal manipulations with the path integral in sections VIA, VIB, including operations with auxiliary topological fields ($b_k(\mathbf{x}), a_k(\mathbf{x})$), are consistent with all general principles. Furthermore, these formal manipulations lead to the correct normalization (47) which we consider as a highly nontrivial consistency check of our path integral approach. We expect that while the $\langle I^2 \rangle_\alpha$ term reproduces the basic normalization, the accounting for auxiliary topological fields (effectively describing the dynamics of vortices) and interactions between them and the Goldstone field should, in principle, generate a nontrivial equation relating n_s and input parameters of the theory. Furthermore, it is quite obvious that the dynamics of the vortices, accounting for their numerous configurations with different densities (affected by the temperature), will also drastically influence the analysis.

Such a computation is obviously a very ambitious goal which is well beyond the scope of the present work. In fact, we do not even attempt to go along this direction as we keep only the quadratic terms in the Hamiltonian, ignoring all the interaction terms, except for long range vortex-vortex interaction proportional to σ . Our goal of the present work is much more modest: we want to demonstrate that our formal procedure leads to a generation of the mass scale, resulting from the auxiliary topological fields. The generation of this scale can be seen already at the lowest quadratic level as we argue below in section VIC. We expect that accounting for the interacting terms using some simplified models such as “vortex ring model” from refs. [9, 10] may renormalize some numerical parameters for the correlation function $\langle I^2 \rangle$. However, we do not expect that the interaction terms may drastically change the basic algebraic structure (51) to be discussed below.

C. Generation of the mass scale

With these comments in mind from previous subsection we now proceed with computations of $\mathcal{I}(a_k)$ contribution defined by (45). The relevant for these computations part of the Hamiltonian (44) includes four-derivative term which requires some extra care. We present $a'_k(\mathbf{x})$ -dependent term in the Hamiltonian as follows

$$\frac{H_{\text{tot}}[a]}{T} = \left(\frac{mT}{2n_s} \right) \int d^3x \frac{a'_k(\mathbf{x})}{2\pi} [\vec{\nabla}^2 \vec{\nabla}^2 - \Delta_\eta^2 \vec{\nabla}^2] \frac{a'_k(\mathbf{x})}{2\pi},$$

$$\Delta_\eta^2 \equiv (2\pi)^2 \left(\frac{\sigma n_s}{mT^2} \right) = \eta \left(\frac{2\pi n_s}{mT} \right)^2, \quad (48)$$

where in the last line we expressed the mass parameter Δ_η in terms of the dimensionless parameter $\eta \sim 1$ replac-

ing the dimensional parameter $\sigma = \eta(n_s/m)$ effectively describing the strength of the vortex-vortex interactions, as discussed after eq. (27). One should comment here that a similar structure with four derivatives also occurs in “deformed QCD” model, see (A15). Therefore, we have some experience how to proceed with computations.

To evaluate the correlation function $\mathcal{I}(a_k)$ defined by eq.(45) we have to find the inverse of the 4-derivative operator entering (48), i.e. $[\vec{\nabla}^2 \vec{\nabla}^2 - \Delta_\eta^2 \vec{\nabla}^2]^{-1}$. To proceed with this task, we use a standard trick to represent the 4-th order operator $[\vec{\nabla}^2 \vec{\nabla}^2 - \Delta_\eta^2 \vec{\nabla}^2]$ as a combination of two terms with the opposite signs. To be more specific, we write

$$\frac{1}{[\vec{\nabla}^2 \vec{\nabla}^2 - \Delta_\eta^2 \vec{\nabla}^2]} = \frac{1}{\Delta_\eta^2} \left(\frac{1}{\vec{\nabla}^2 - \Delta_\eta^2} - \frac{1}{\vec{\nabla}^2} \right), \quad (49)$$

such that the Green’s function for $a'_k(\mathbf{x})$ field which enters the expression for the winding susceptibility (45) can be represented as a combination of two Green’s functions: conventional massive field and the “ghost-like” field with “wrong” sign. Naively, the presence of 4-th order operator in eq. (48) is a signal that the “ghost-like” instability develops with violation of the unitarity and causality occurring in the system. However, this naive suspicion is obviously incorrect as the Hamiltonian (48) was derived from the original perfectly defined and unitary QFT. Technically, the absence of any fundamental deficiencies in our system is explained by the fact that $a'_k(\mathbf{x})$ field is not a dynamical field with canonical kinetic term. Instead, this field is an auxiliary topological field which does not propagate as a conventional dynamical degree of freedom.

A very similar structure (A17) is known to occur in strongly coupled QCD. The corresponding topological field in QCD is known as the Veneziano ghost. Precisely this structure is the key element in resolution of the celebrated $U(1)$ problem when the $U(1)$ singlet Goldstone field receives its mass as a result of mixing of would be Goldstone field with the topological Veneziano ghost. We elaborate on this very close analogy in Appendix A and also in next section VIII with specific comments on similarities and differences between the the winding number susceptibility $\langle I^2 \rangle$ in superfluid system and topological number susceptibility $\langle Q^2 \rangle$ in confined QCD. In fact our notation for Δ_η in eq. (48) is inspired by this amazing analogy.

We now proceed with explicit computations. The relevant correlation function which enters the expression for the winding susceptibility (45) can be explicitly com-

puted using expression (49) for inverse operator as follows

$$\begin{aligned}
\mathcal{I}(a_k) &= - \left(\frac{mT}{2\pi n_s} \right)^2 \frac{\int \mathcal{D}[a'] e^{-\frac{H[a']}{T}} \vec{\nabla}^2 a'_k(\mathbf{x}), \vec{\nabla}^2 a'_k(\mathbf{x}')}{\int \mathcal{D}[a'] e^{-\frac{H[a']}{T}}} \\
&= -3 \left(\frac{mT}{n_s} \right) \int \frac{d^3 p}{(2\pi)^3} e^{i\mathbf{p} \cdot (\mathbf{x} - \mathbf{x}')} \frac{p^4}{\Delta_\eta^2} \left[-\frac{1}{p^2 + \Delta_\eta^2} + \frac{1}{p^2} \right] \\
&= -3 \left(\frac{mT}{n_s} \right) \int \frac{d^3 p}{(2\pi)^3} e^{i\mathbf{p} \cdot (\mathbf{x} - \mathbf{x}')} \left[\frac{p^2}{p^2 + \Delta_\eta^2} \right] \\
&= -3 \left(\frac{mT}{n_s} \right) \left[\delta(\mathbf{x} - \mathbf{x}') - \Delta_\eta^2 \frac{e^{-\Delta_\eta |\mathbf{x} - \mathbf{x}'|}}{4\pi |\mathbf{x} - \mathbf{x}'|} \right], \quad (50)
\end{aligned}$$

where coefficient 3 in front is due to three different components $a'_k(\mathbf{x})$ equally contributing to $\mathcal{I}(a_k)$. The corresponding contribution to $\langle I^2 \rangle_a$ after integration over volume $\int d^3 x d^3 x'$ is

$$\begin{aligned}
\langle I^2 \rangle_a &= \int d^3 x \int d^3 x' \cdot \mathcal{I}(a_k) = \\
&-3 \left(\frac{mT}{n_s} \right) \int d^3 x \int d^3 x' \left[\delta(\mathbf{x} - \mathbf{x}') - \Delta_\eta^2 \frac{e^{-\Delta_\eta |\mathbf{x} - \mathbf{x}'|}}{4\pi |\mathbf{x} - \mathbf{x}'|} \right]. \quad (51)
\end{aligned}$$

Now we are in position to make few important comments on the obtained results. First of all, formula (50) has the structure which is identical to (A16) derived for the “deformed QCD” model, where independent computations support this formal manipulations with topological auxiliary fields. Therefore, we are quite confident in our technique.

Now, the contact term proportional to the delta function in eq. (51) exactly cancels the contact term (47) generated by the Goldstone field such that the formula for $\langle I^2 \rangle$ assumes the form

$$\begin{aligned}
\langle I^2 \rangle &= \langle I^2 \rangle_\alpha + \langle I^2 \rangle_a + (\text{other terms}) \\
&= 3 \left(\frac{mTV}{n_s} \right) \int_{R_0} d^3 x \left(\Delta_\eta^2 \frac{e^{-\Delta_\eta r}}{4\pi r} \right) + (\text{others}), \quad (52)
\end{aligned}$$

where we introduced the cutoff parameter R_0 to parametrize the deviation for the vortex-vortex interaction from the ideal case of infinitely thin vortices. The terms indicated as “others” in eq. (52) account for all types of interactions which were consistently ignored in our simplified treatment as we are dealing with the quadratic terms in the Hamiltonian only. We want to rewrite our expression (52) in the following form which is quite convenient for our numerical estimates in next section VII,

$$\begin{aligned}
\langle I^2 \rangle &= 3 \left(\frac{mTV}{n_s} \right) f(z) + (\text{others}), \quad (53) \\
f(z) &\equiv \int_{R_0} d^3 x \left(\Delta_\eta^2 \frac{e^{-\Delta_\eta r}}{4\pi r} \right), \quad z \equiv \Delta_\eta R_0,
\end{aligned}$$

where dimensionless function $f(z=0) = 1$ is normalized to one at $z = 0$, and becomes exponentially small at large z . One should emphasize that both dimensional

parameters, Δ_η and cutoff scale R_0 (which represents an effective size of a vortex core) are obviously a nontrivial functions of density n_s and temperature T , as one can see from definition (48). Therefore, z is also a nontrivial parameter of the density n_s and temperature. One can argue that z in vicinity of the phase transition scales as $z \sim \tau^{-\beta/2}$, see eq. (55) for the definition of the reduced temperature τ .

The most important result of this subsection is an explicit demonstration that the system generates the mass scale as a result of vortex-vortex interaction. Such an interpretation of the obtained gap (resulting from vortex-vortex interaction) is motivated by the expression (48) for the mass Δ_η which is explicitly proportional to the strength of the vortex-vortex interaction. It is naturally to assume that the corresponding scale Δ_η is related to some elementary excitations, see few comments on this assumption in next section. However, the corresponding excitation gap Δ_η cannot be identified (due to a number of reasons) with the well known roton’s gap ϵ_0 defined by eq.(9). A simplest argument supporting this claim is that our gap $\Delta_\eta \sim n_s(T)$ vanishes at $T \rightarrow T_c$ in contrast with roton’s gap which remains finite in vicinity of the phase transition at $T \simeq T_c$ as reviewed in section III.

VII. THE VORTONS

In this section we want to elaborate on the nature of the gap Δ_η and make few simple numerical estimates for these new topological objects. They will be dubbed the *vortons* in what follows. The corresponding gap in vicinity of the phase transition can be approximated as follows

$$\Delta_\eta(T \rightarrow T_c) \simeq \sqrt{\eta} \left(\frac{2\pi n}{mT} \right) \cdot \left(\frac{T_c - T}{T_c} \right)^{2\beta}, \quad \beta = \frac{1}{2}, \quad (54)$$

where we assume that the critical exponent $\beta = 1/2$ as in conventional Landau-Ginsburg approximation¹. We will provide some justification for the term “vorton” later in the text. But first, we want to explain some key properties of these unusual objects.

Most important feature of the vorton is its T -dependent gap (54). Furthermore, this object completely disappears from the system in normal phase at $T > T_c$ because it exists in the system only when superfluid vortices exist. The vortons always accompany the vortices and disappear from the system together with the superfluid vortices at $T > T_c$.

One can numerically estimate a typical gap for vortons in vicinity of the critical temperature when T is close to

¹ Experimentally $\beta \simeq 1/3$ rather than $1/2$. The difference is normally explained using renormalization group procedure. We consistently ignore all the interacting terms in our analysis. Therefore, we use “bare value” $\beta = 1/2$ in our estimates.

T_c as follows,

$$\Delta_\eta(T \rightarrow T_c) \sim 10K\sqrt{\eta} \cdot \tau^{2\beta}, \quad \tau \equiv \left(\frac{T_c - T}{T_c} \right). \quad (55)$$

Numerically², the gap Δ_η is very similar to the roton's gap $\epsilon_0 \simeq 8K$ away from T_c . However, the vorton's gap Δ_η becomes much smaller than the roton's gap at $(T_c - T) \ll T_c$ when the vortons become light³.

If one assumes that dimensionless parameter η , effectively describing the interaction between vortices (which themselves become “fat” when the core of a vortex and its length start to scale in a similar way in vicinity of the phase transition [10]) is proportional to the overlapping volume between two vortices than η in vicinity of the phase transition scales as $\eta \sim \tau^{-3\beta}$ while the gap $\Delta_\eta \sim \tau^{\beta/2}$. If one also assumes that the “other” terms in (52) do not qualitatively change the physics, the winding number correlation function $\langle I^2 \rangle$ vanishes at the critical temperature as $\langle I^2 \rangle \sim \exp(-\tau^{-\beta/2})$. These estimates should be taken with great precaution and grain of salt because it is very hard to justify any of these assumptions.

It is quite obvious that the vorton's excitations can be completely ignored far away from the phase transition. First, they become very heavy as one can see from T -dependent formula for the gap (54), and therefore, they can be excited only in a close vicinity of $T \simeq T_c$. In fact, this is the main reason for our approximate expression (55). The second reason for an additional suppression of vortons at low $T \ll T_c$ is the observation that the vortons are intimately related to vortices as they obviously contribute to the winding number correlation function $\langle I^2 \rangle$ according to (52). Their contribution to other correlation function which does not include the vorticity operator is suppressed, see few comments below. Therefore, when the network of the vortices is not yet formed, or not well-developed at sufficiently low $T \ll T_c$ the vortons are also absent in the system.

Another key characteristic of the vorton is its spin. Vorton is described in terms of the (axial) vector topological field $a_k(\mathbf{x})$ which corresponds to $S = 1$. The coefficient 3 in front of the correlation function (50) can be interpreted as the degeneracy factor $(2S + 1)$. This coefficient 3 is precisely what is required for the cancellation of the contact terms as expressed by eq.(52).

One may wonder: why and how the topological field $a_k(\mathbf{x})$ which was originally introduced into the partition function (32), (38) as an auxiliary topological vector field becomes a dynamical degree of freedom? The answer is that this auxiliary field $a_k(\mathbf{x})$ is mixing with the physical propagating Goldstone field $\alpha(\mathbf{x})$ in a course of diagonalization of the Hamiltonian (44) according to (43). Precisely this mixing eventually makes the auxiliary field (some part of it) $a_k(\mathbf{x})$ to become a physical quasiparticle-like degree of freedom. At the same time, another portion of the $a_k(\mathbf{x})$ field remains a non-propagating degree of freedom and manifests itself as a contact term $\sim \delta(\mathbf{x} - \mathbf{x}')$ contributing to the correlation function (51) with the “wrong sign” which is opposite to conventional contribution of the physical Goldstone field (46), (47). This “wrong sign” provides a specific mechanism of removal of the winding number from the system which happens at the phase transition point T_c when this removal is completed.

One can view formula (51) as an explicit manifestation of the “dual” nature of the topological field $a_k(\mathbf{x})$: the first contact term $\sim \delta(\mathbf{x} - \mathbf{x}')$ is obviously not related to any propagating degrees of freedom, while the second term is obviously related to a physical propagating degree of freedom, the vorton, with a gap Δ_η .

Such a “dual” behaviour of the auxiliary topological field once again amazingly resembles the behaviour of the QCD Veneziano ghost which contributes with a “wrong” sign to the QCD topological susceptibility $\langle Q^2 \rangle$, mixes with the Goldstone field and becomes the massive and propagating η' field. This resolution of the renowned $U(1)$ problem as formulated by Witten and Veneziano in 1980 is well supported by a numerous lattice simulations and commonly accepted by the QCD community, see Appendix A for details and references.

It is instructive to compute the correlation function $\langle a'_i(\mathbf{x}), a'_j(\mathbf{x}') \rangle$ itself for a better understanding of the nature of the topological auxiliary field. This computation will also provide a much closer analogy with Veneziano ghost mentioned above. To accomplish this task we define the correlation function $\mathcal{V}_{ij}(a_k)$ which is directly related to the previously computed correlation function $\mathcal{I}(a_k)$ as follows,

$$\mathcal{V}_{ij}(a_k) = - \left(\frac{mT}{n_s} \right)^2 \left\langle \frac{a'_i(\mathbf{x})}{2\pi}, \frac{a'_j(\mathbf{x}')}{2\pi} \right\rangle, \quad (56)$$

$$\mathcal{I}(a_k) = \vec{\nabla}_{\mathbf{x}}^2 \vec{\nabla}_{\mathbf{x}'}^2 \mathcal{V}_{ii}(a_k)$$

where $\mathcal{I}(a_k)$ is given by (50) and (51). The computation of the correlation function $\mathcal{V}_{ij}(a_k)$ can be carried out in a similar manner with result

$$\mathcal{V}_{ij}(a_k) = - \left(\frac{mT}{2\pi n_s} \right)^2 \frac{\int \mathcal{D}[a'] e^{-\frac{H[a']}{T}} a'_i(\mathbf{x}), a'_j(\mathbf{x}')}{\int \mathcal{D}[a'] e^{-\frac{H[a']}{T}}} \quad (57)$$

$$= \delta_{ij} \left(\frac{mT}{\Delta_\eta^2 n_s} \right) \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p} \cdot (\mathbf{x} - \mathbf{x}')} \left[\frac{1}{p^2 + \Delta_\eta^2} - \frac{1}{p^2} \right].$$

The structure of this expression is very suggestive. First of all, the physical contribution with the gap $\sim \Delta_\eta$ is pre-

² We expect that these numerical estimates change as a result of many effects mentioned at the end of subsection VIB, and which were completely ignored in present analysis. However, we do not expect that the fundamental properties of the system may drastically change as a result of these simplifications.

³ It is interesting to note that a quasiparticle with such unusual dispersion relation (55) when the gap vanishes at $T = T_c$ has been postulated in [8] and dubbed as “helon”. We cannot comment on relation (if any) between the two objects as the framework and technique developed in sections VIB, VIC is drastically different from ref. [8].

cisely the contribution of vortons in our previous computations of the winding number correlation function (50) and (51). The pole at $p^2 = 0$ with a “wrong sign” which is present in (57) might look very suspicious and dangerous. However, this pole is present in gauge-dependent correlation function $\mathcal{V}_{ij}(a_k)$. This pole generates the contact term $\delta(\mathbf{x}-\mathbf{x}')$ in gauge invariant correlation function (50) and (51) which is obviously consistent with all fundamental principles of the theory.

This pole does not contribute to any other correlation functions, with exception of the winding number susceptibility $\langle I^2 \rangle$ where it manifests itself as the contact term. In other words, the strange pole does not correspond to any physical massless degrees of freedom as its only role is the generation of the contact term in the winding number correlation function $\langle I^2 \rangle$, similar to QCD case when the Veneziano ghost contributes exclusively to $\langle Q^2 \rangle$ and nowhere else. Nevertheless, this pole at $p^2 = 0$ explicitly shows that the physics behind the contact term in gauge invariant correlation function (50) is due to the infrared (IR) rather than ultraviolet (UV) physics, which is very similar to QCD where one can show that the contact term in $\langle Q^2 \rangle$ is saturated by IR rather than by UV physics.

As we review in Appendix A the direct analog of eq.(57) is eq. (A19) in “deformed QCD” model. Furthermore, one can explicitly construct the physical Hilbert space in a such a way that the unphysical ghost-like particles are removed from the physical amplitudes by imposing the Gupta-Bleuler-like condition on the physical Hilbert space, see (A24). We shall not elaborate on this topic in the present work, by mentioning that the pole in (57) is really harmless with the “only” trace that it contributes to the contact term with the “wrong” sign and has the tendency to cancel the winding number which is present in the system. This tendency becomes a much more profound phenomenon in vicinity of the phase transition when the original winding number vanishes at $T = T_c$ precisely as a result of this cancellation, and the system becomes a conventional non-superfluid liquid at $T \geq T_c$.

From these discussions it is quite obvious that the manifestations of the vortons are drastically different from conventional observables resulted from fluctuations of “normal” degrees of freedom such the rotons and phonons. In particular, the vortons cannot be observed as the pole in the well studied “density-density” response function defined by (8), (9) simply because the topological field $a_k(\mathbf{x})$ does not couple directly to the density fluctuations. Instead, it couples to the circulations $\gamma_k(\mathbf{x})$ as eq.(28) states. It also explains why the vorton excitations do not manifest themselves in conventional inelastic X-ray and neutron scattering experiments. Rather, the vortons contribute to the winding number susceptibility $\mathcal{I}(a_k)$ according to (50). The interaction of the vortons with other quantum fluctuations (which can be in principle computed using renormalization group procedure) should generate some contribution to the “density-density” response function as well. However, we expect

this contribution to be numerically strongly suppressed due to a small mixing of the topological field $a_k(\mathbf{x})$ with the Goldstone field (43) at $T \simeq T_c$ when the vortons are sufficiently light (and therefore, can be easily excited), however the superfluid density is already quite small $n_s \sim (T_c - T)^{2\beta}$ at $T \simeq T_c$. As we mentioned previously, any effects of vortons at very low $T \ll T_c$ are exponentially small and can be safely ignored.

Our next comment in this subsection is devoted to the term “vorton”. We want to justify our terminology by dubbing the topological excitations $a_k(\mathbf{x})$ emerging in our framework as “vortons”. Originally, the vortons were introduced in particle physics (relativistic systems) when a superconducting string may form a macroscopically large closed stable vortex loop. The stabilization occurs due to condensation of another field in the vortex core and non-dissipating current moving along the core of the vortex. Similar classical objects were also constructed in non-relativistic systems with Bose-Einstein Condensates, see [12] for references and details. In particular, it has been shown in [12] that the vorton carries the angular momentum, and the stability of the vortons can be understood in terms of conservation of angular momentum.

Our elementary objects with typical microscopical scales (55) are obviously quantum, not classical, objects. Nevertheless, they have a number of common features which motivated our terminology in the present work. Indeed, the circulation field plays a key role in classical as well in quantum description of the vortons represented by $a_k(\mathbf{x})$ in our construction. Furthermore, the role of non-dissipating conserved current in construction of classical object plays the circulation source $\gamma_k(\mathbf{x})$ in present work. In addition, both objects, classical and quantum, are characterized by angular momentum which is the spin $S = 1$ in our construction of quantum vortons. Finally, the vortons in our framework explicitly contribute to the correlation function (50), which is analogous to the computation of the integer winding number in construction of the classical configuration [12]. Based on these similarities one can view the quantum elementary vorton described by $a_k(\mathbf{x})$ in our construction as a microscopical elementary ring of a circulating conserved current $\gamma_k(\mathbf{x})$. These objects represent the counterparts of the macroscopically large classical vortons. Such a view is, in fact, very close to Feynman’s interpretation of relevant quasi-particles responsible for the breakdown of the superfluidity [7].

• We conclude this section with the following generic remark. The main claim of this section is that the long range physics due to the topological configurations (in form of a complicated network of knotted, folded and wrinkled vortices) can be formulated in terms of an auxiliary topological fields $[a_k(\mathbf{x}), b_k(\mathbf{x})]$. These fields do not propagate. Rather, they effectively describe the long-range dynamics of this complicated network of vortices. Though these fields are not independent degrees of freedom, they still produce a number of physical effects. In particular, $a_k(\mathbf{x})$ mixes with the Goldstone field, be-

comes a massive propagating degree of freedom, the vorton. The gap (54) of the vorton has a unique feature: it vanishes at the phase transition temperature T_c . Furthermore, the vorton disappears from the spectrum at $T \geq T_c$.

The topological fields $[a_k(\mathbf{x}), b_k(\mathbf{x})]$, being the emergent gauge fields, do not completely decouple from the gauge invariant observables. In particular, they do contribute to the winding number correlation function $\langle I^2 \rangle$ due to its unique structure. The emergent mass scale (54) manifests itself precisely by studying this specific correlation function. It is a highly nontrivial phenomenon, and it has its counterpart in relativistic particles physics, where a similar new scale is generated in very similar manner. This novel scale is parametrically suppressed $\sim 1/N$ in large N QCD, and it can be explicitly tested by studying the topological susceptibility correlation function, $\langle Q^2 \rangle$. Furthermore, the Veneziano ghost which plays the same role as auxiliary topological fields $[a_k(\mathbf{x}), b_k(\mathbf{x})]$ in our present construction, generates the contact term in $\langle Q^2 \rangle$. Precisely this contact term is the crucial element in the resolution of the celebrated $U(1)$ problem. The Veneziano ghost does not contribute to any other correlation functions. We review this well established mechanism in a simplified “deformed QCD” model in Appendix A with emphasize on analogy and similarities with topological features of the superfluid systems studied above.

VIII. QCD VS. SUPERFLUIDITY: $\langle Q^2 \rangle$ VS. $\langle I^2 \rangle$

In this section we want to discuss a number of similarities and differences between our computations $\langle I^2 \rangle$ for superfluid liquid and the computations of $\langle Q^2 \rangle$ in QCD. First, we list a number of formal similarities in subsection VIII A. In subsection VIII B we formulate the fundamental difference between the two systems. Finally, in subsection VIII C we compare the long range structure observed in lattice simulations in strongly coupled QCD with vortex network studied in Helium II experiments.

A. Formal similarities

We already mentioned a number of formal similarities in correlation functions $\langle I^2 \rangle$ for superfluid system and $\langle Q^2 \rangle$ for “deformed QCD” model. We want to list them for convenience once again. The role of eq.(38) plays (A12) in “deformed QCD” model. The equations (41), (42) in superfluid system (which includes mixing with the Goldstone field) are formally similar to (A13). Four derivative operator (48) in superfluid system is identically the same as 4-derivative operator (A15) which occurs in “deformed QCD” model. Finally, $\langle I^2 \rangle_a$ has the structure (50) which is very much the same as $\langle Q^2 \rangle$ given by eq.(A16) in the QCD like model. The topological auxiliary fields $[a_k(\mathbf{x}), b_k(\mathbf{x})]$ in superfluid system play the

same role as $[a(\mathbf{x}), b(\mathbf{x})]$ in “deformed QCD” model.

One should comment here that we refer in this “formal similarities” section to the weakly coupled gauge theory, the “deformed QCD” model (where analytical computations can be explicitly performed) rather than to strongly coupled QCD. However, it is expected that very similar structure in topological susceptibility $\langle Q^2 \rangle$ also occurs in real strongly coupled QCD, see [15] and references therein. Furthermore, it has been argued [13], that there is no any phase transition along the passage from weakly coupled “deformed QCD” model to strongly coupled QCD.

B. Fundamental difference between QCD and superfluid systems

These formal similarities, however, should not hide a fundamental difference between superfluid systems defined in Minkowski space time and Euclidean 4d “deformed QCD” model which is reviewed in Appendix A. In particular, instead of real vortices in superfluid system we have pseudo-particles (monopoles) which saturate the path integral. These pseudo-particles describe the time-dependent tunnelling transitions between physically identical but topologically distinct winding $|n\rangle$ states connected by the large gauge transformations. These pseudo-particles are not real physical states in Hilbert space, in contrast with vortices in a superfluid system. It means that these pseudo-particles should be treated as a convenient computational technical tool in evaluation of the path integral. It should be contrasted with quasi-particles (phonons and rotons) in superfluid systems which can propagate in physical space-time, and which can be explicitly observed by studying a varies correlation functions.

The observation of real quasi-particles and real vortices in a number of different physical experiments, see textbook [3] for references, should be contrasted with observed topological structure in the path integral computations in QCD using the lattice simulations in 4d Euclidean space-time, see next section VIII C for the details and references.

However, there is a common denominator between these two very different studies (superfluidity vs QCD) carried out in two very different space-times (Minkowski vs Euclidean). In both cases there is a hidden long range order. It is formally expressed in terms of a non-physical pole at $p^2 = 0$ in eqs.(57) and (A19) in gauge-dependent correlation functions. This pole, being unphysical itself, nevertheless generates the physical contact terms in gauge-invariant correlation functions $\langle I^2 \rangle$ and $\langle Q^2 \rangle$ correspondingly. The manifestations of this long range structure is very different for two systems: in superfluid systems this long range structure manifests itself as a presence of a network of long vortices with a typical length scale of order of the size of a system [20], while in QCD this long range structure can be observed in lattice

simulations where relevant 4D topological configurations are correlated on scales of order the the entire lattice [21]. We elaborate on this similarity in next subsection VIII C.

C. Network of superfluid vortices in HeliumII [20] vs “Skeleton network” in lattice QCD [21]

Before we elaborate on connection between the network of superfluid vortices (unambiguously observed in Helium [20]) and a complicated topological structure in QCD (observed in Monte Carlo lattice simulations, dubbed as “skeleton network” [21]), we want to list the main properties of the “skeleton network”.

The gauge configurations observed in [21] display a laminar structure in the vacuum consisting of extended, thin, coherent, locally low-dimensional sheets of topological charge embedded in 4d space, with opposite sign sheets interleaved, see original QCD lattice results [21–24]. A similar structure has been also observed in QCD by different groups [25–29]. Furthermore, the studies of localization properties of Dirac eigenmodes have also shown evidence for the delocalization of low-lying modes on effectively low-dimensional surfaces. The following is a list of the key properties of these gauge configurations which we wish to review:

- 1) The tension of the “low dimensional objects” vanishes below the critical temperature and these objects percolate through the vacuum, forming a kind of a vacuum condensate;
- 2) These “objects” do not percolate through the whole 4d volume, but rather, lie on low dimensional surfaces $1 \leq d < 4$ which organize a coherent double layer structure;
- 3) The total area of the surfaces is dominated by a single percolating cluster of “low dimensional object”;
- 4) The contribution of the percolating objects to the topological susceptibility $\langle Q^2 \rangle$ has the same sign compared to its total value;
- 5) The width of the percolating objects apparently vanishes in the continuum limit;
- 6) The density of well localized 4d objects (such as small size instantons) apparently vanishes in the continuum limit.

This structure obviously collapses above the phase transition point at $T > T_c$. These drastic changes at the critical temperature can be, in fact, quantitatively understood [30] in large N limit when the topological susceptibility is saturated by conventional instantons at $T > T_c$, while it is presumably saturated by the “skeleton network” at $T < T_c$ described above⁴.

It has been argued for quite a long time, see [3] for references, that the microscopical origin of the phase transition in superfluid systems is somehow related to the network of vortices. This coherent network is expected to become very dense, highly knotted, folded and crumpled when the critical temperature approaches the phase transition point. The superfluid vortices, which represent the main constituent of this network, of course disappear above T_c . Entire network is expected to collapse exactly at this point as the building material, the superfluid vortices disappear at $T = T_c$. Details of the dynamics describing this complicated picture of the network’s evolution is obviously prerogative of numerical simulations, similar to the QCD studies [21–28].

We would like to speculate here that the structure which has been observed on the lattice [21–28] plays the same role as the network of vortices experimentally observed in superfluid He II [20]. In other words, we want to advocate an idea that the corresponding networks provide the microscopical mechanisms responsible for the phase transitions in both cases: confinement-deconfinement phase transition in the QCD case, and superfluid to the normal liquid phase transition in case of superfluid helium II.

Indeed, in both cases the configurations themselves have lower dimensionality than the space itself. However, these low-dimensional configurations are so dense, and they fluctuate so strongly that they almost fill the entire space. In both cases an effective tension (representing the superposition of internal tension combined with the entropy) of the configurations vanishes as a result of large entropies of the objects which overcome the internal tension. This leads to the percolation of the vortices in superfluid He II and formation of the the “Skeleton” in QCD correspondingly. If the effective tensions of these configurations did not vanish, we would observe a finite number of fluctuating objects with finite size in the system instead of observed percolation of the “Skeleton” and superfluid vortices. Furthermore, typically the “Skeleton” spreads over maximal distances percolating

is also known that the “deformed QCD” model becomes the strongly coupled QCD when the size of the compact S^1 adiabatically increases. The conjecture [31] is that some extended topological objects which are inevitably present in the “deformed QCD” model, eventually form the “skeleton network” observed in [21–28] when one slowly moves from weakly coupled model to strongly coupled QCD by increasing the size of S^1 . The structure of these objects obviously must drastically change in passage from weakly coupled to strongly coupled regime. Recent numerical studies [32, 33] of a similar model (where point-like fractionally charged monopoles saturate the partition function) implicitly support this conjecture. Indeed, the semiclassical approximation employed in [32, 33] breaks down when the size of S^1 increases. It can be interpreted as that the elementary point-like monopoles cannot saturate the partition function at sufficiently large size of S^1 and must transform themselves into a more complicated (extended) objects. The dynamics of this transformation (from point-like pseudoparticles to extended structure) is still not well understood.

⁴ One may wonder how the “skeleton network” at $T < T_c$ can emerge from the fractionally charged point-like monopoles of the “deformed QCD” model? Indeed, it is known that all important features of the “deformed QCD” model are saturated by fractionally charged monopoles, see Appendix A for review. It

through the entire volume of the system similar to superfluid vortices.

Our final comment is related to the \mathcal{P} invariance in both systems. The question on \mathcal{P} invariance occurs in QCD due to the fact that the topological density operator Q is a pseudoscalar. A similar comment also holds for circulation density field $\gamma_k(\mathbf{x})$ defined by eq. (22) which is the axial rather than vector field. Our auxiliary topological field $a_k(\mathbf{x})$ is also axial rather than vector field as the relation (24) states. The correlation functions $\langle I^2 \rangle_a$ is defined in terms of $a_k(\mathbf{x})$ by eq. (50), while $\langle Q^2 \rangle$ in “deformed QCD” model is defined in terms of $a(\mathbf{x})$ by (A13). These correlation functions are obviously \mathcal{P} -even observables constructed from \mathcal{P} -odd objects. In “Skeleton” studies [21] there are two oppositely-charged sign-coherent connected structures (sheets). The \mathcal{P} invariance holds in QCD as a result of delicate cancellation between the opposite sign interleaved sheets.

A similar studies on sign of the circulation in superfluid He II have not been performed in [20]. However, it is quite obvious that the \mathcal{P} invariance can be locally maintained only as a result of similar cancellation between opposite sign coherent interleaved vortices with opposite circulations.

One should remark here that if the external parameter $\theta \neq 0$ is not vanishing in QCD than the delicate cancellation mentioned above does not hold anymore, and \mathcal{P} asymmetry would be generated in the system⁵. Similar comment also applies to a superfluid system when the role of θ plays external parameter \vec{s} defined by eqs. (13), (14). Indeed, up to normalization factors both correlation functions $\langle I^2 \rangle$ and $\langle Q^2 \rangle$ are expressed as the second derivatives of the partition function \mathcal{Z} with respect to \vec{s} and θ correspondingly

$$\langle I^2 \rangle \sim \frac{\partial^2 \ln \mathcal{Z}}{\partial \vec{s}^2}, \quad \langle Q^2 \rangle \sim \frac{\partial^2 \ln \mathcal{Z}}{\partial \theta^2}. \quad (58)$$

It is quite obvious that $\vec{s} \neq 0$ in superfluid liquid breaks \mathcal{P} invariance of the system (similar to QCD at $\theta \neq 0$) as it corresponds to a presence of superfluid flow in a specific direction according to (13).

The crucial difference between the two networks discussed above is of course the nature of the constituents of these networks: superfluid macroscopically large vortices live in real Minkowski space-time while lattice QCD measurements are done in Euclidean space-time where the corresponding long ranged configurations saturate the path integral, and describe the tunnelling processes, as we already discussed in section VIII B.

IX. CONCLUSION. SPECULATIONS.

Our conclusion can be separated into three related, but still distinct pieces: First, we highlight the theoretical results on computations of $\langle I^2 \rangle$ based on the path integral approach in a superfluid system. Secondly, we mention on possible connection of our computations with related works in other fields. Finally, we present some speculations related to strongly coupled QCD where fundamentally the same effects do occur, and might be the crucial ingredients in understanding of the observed cosmological vacuum energy today, the so-called dark energy.

A. Results.

The main “technical” claim of this work is that the contribution of a complicated network of knotted, folded, twisted and wrinkled vortices to winding number susceptibility $\langle I^2 \rangle$ can be formulated in terms of an auxiliary topological fields $[a_k(\mathbf{x}), b_k(\mathbf{x})]$. These fields are not new degrees of freedom, and they do not propagate. Rather, they effectively describe the long-range dynamics of this complicated network of strongly interacting vortices. Though these fields are not independent degrees of freedom, they still produce a number of physical effects. In particular, they generate the contact term in the winding number correlation function $\langle I^2 \rangle$. In addition, $a_k(\mathbf{x})$ mixes with the Goldstone field, becomes a massive propagating degree of freedom, the vorton.

It is a highly nontrivial phenomenon, and it has its counterpart in relativistic particles physics, where a similar effect does occur, and it represents the resolution of the celebrated $U(1)$ problem as formulated by Witten and Veneziano in 1980 [16, 17]. Precisely the Veneziano ghost, which plays the role of topological auxiliary fields $[a_k(\mathbf{x}), b_k(\mathbf{x})]$ in present context, is the crucial element in the resolution of the $U(1)$ problem and generates the contact term in topological susceptibility $\langle Q^2 \rangle$ confirmed by numerous lattice simulations. The Veneziano ghost does not contribute to any other correlation functions. We have made a number of comments demonstrating a close relation between analysis of $\langle I^2 \rangle$ in superfluid system (discussed in sections VI, VII) and $\langle Q^2 \rangle$ in a simplified “deformed QCD” model reviewed in Appendix A.

It might look very suspicious that a complicated dynamics of network of the vortices (which is known to emerge in superfluid systems in vicinity of the phase transition) can be described in so simple way in terms of local auxiliary topological fields $[a_k(\mathbf{x}), b_k(\mathbf{x})]$. However, such a complimentary description becomes less mysterious (but still highly nontrivial) if one recalls that many systems demonstrate the particle-vortex duality which has been known since [34]. Therefore, it is not really a big surprise that some elements of an approximate “duality” (between superfluid vortices and local topological fields $[a_k(\mathbf{x}), b_k(\mathbf{x})]$) also emerge in our superfluid system.

It is quite obvious that our studies of $\langle I^2 \rangle$ in terms

⁵ This is renowned strong \mathcal{CP} problem in QCD as the θ term violates both: \mathcal{P} and \mathcal{CP} symmetries of the theory. The resolution of this problem consists a new fundamental particle, the axion, yet to be discovered, which dynamically drives $\theta \rightarrow 0$ during the QCD phase transition in early Universe.

of topological auxiliary fields $[a_k(\mathbf{x}), b_k(\mathbf{x})]$ as advocated in the present work is only the very first step in direction of complete description of $\langle I^2 \rangle$ which is precisely related to superfluid density as thermodynamical relation (14) states. Indeed, we kept only the quadratic terms in the Hamiltonian by consistently neglecting all the interactions in order to compute the path integral. Accounting for these interactions is obviously very ambitious task which is well beyond the scope of the present work. However, we expect (based on experience with “deformed QCD” model) that the corresponding interactions would “renormalize” the parameters of the system, but not drastically change the structure of the winding number correlation function (50), including the contact term, which was computed by keeping the dominant quadratic terms and neglecting all sub-leading (at large distances) interacting terms⁶.

B. Relations to other approaches.

We want to make few additional comments on possible relation of our treatment of $\langle Q^2 \rangle$ with other computations where similar structure for topological susceptibility $\langle Q^2 \rangle$ is known to occur. First of all we have in mind the saturation of the topological susceptibility $\langle Q^2 \rangle$ by the Veneziano ghost [17] which can be understood in terms of auxiliary topological fields $[a(\mathbf{x}), b(\mathbf{x})]$ in our computations as reviewed in Appendix A. The key observation relevant for the present work is that the corresponding contact term must be related somehow to the so-called Gribov’s copies [35]. Indeed, in the Coulomb gauge the emergence of the Gribov’s copies can be traced to the presence of the topological sectors in a gauge theory. The tunnelling transitions between these topological sectors saturate the contact term in topological susceptibility as reviewed in Appendix A. Therefore, the contact term, which is the key element in our analysis (based on formulation in terms of the auxiliary topological fields) must be related to the Gribov’s copies which describe this type of physics.

Furthermore, it has been recently argued that the Gribov’s copies are also inherently related to confinement in QCD, see [36] and references on related works therein. Such a relation strongly suggests that the auxiliary topological fields (which is the main technical tool in our approach) are ultimately related to the nature of the confinement of the theory as they saturate the topological susceptibility $\langle Q^2 \rangle$.

In the “deformed QCD” model reviewed in Appendix A all these relations and connections is almost a trivial remark as the topological auxiliary fields effectively describe the dynamics of the monopoles which indeed pro-

vide the confinement in the theory. However, our claim is much more generic and applies to the strongly coupled gauge theory where explicit relation between the two descriptions is far from obvious. The phase transition formulated in terms of the local auxiliary fields implies that these topological fields experience the drastic modifications in vicinity of the phase transition.

In context of the present work where the phase transition in He-II can be understood in terms of the percolating network of vortices the corresponding drastic changes indeed take place in the system as we discussed in section VII. If we assume that the analogy between superfluid He II and the QCD vacuum is sufficiently deep, as argued in section VIII then one may ask the following question: what kind of objects play the role of superfluid vortices in the QCD phase transition? We obviously do not know a precise answer to this question. However, we strongly suspect that the centre vortices, see [37] for review, which have been observed in numerous lattice simulations (and which generate the dominant contribution to the string tension and topological susceptibility in the confined phase) may play the role of superfluid vortices responsible for the phase transition in He-II.

The dual description in both cases is formulated in terms of local auxiliary topological fields: $[a(\mathbf{x}), b(\mathbf{x})]$ in “deformed QCD” (and corresponding generalization in terms of the Veneziano ghost in strongly coupled QCD), and $[a_k(\mathbf{x}), b_k(\mathbf{x})]$ in superfluid system. The corresponding similarities and analogies between the two systems have been extensively discussed in section VI. The descriptions of the two systems in terms of the original variables (superfluid vortices in He-II and, possible centre vortices in QCD) most likely are very different; it is just the dual descriptions formulated in terms of the auxiliary topological fields in two drastically different systems look very much the same.

C. Speculations.

The final part of our conclusion is much more speculative, but we think it is worthwhile to mention these speculations because they may have profound consequences on our understanding of the nature of the vacuum energy of the Universe we live in.

The key point of our analysis is the presence of the IR pole at $p^2 = 0$ in gauge dependent correlation functions in superfluid system (57) and analogous relation (A19) in QCD. While there are no any physical massless degrees of freedom associated with this pole, it still generates a physical contribution to gauge invariant correlation functions in form of the contact terms (50) and (A16) in superfluid liquid in QCD correspondingly. The crucial point is that the corresponding contact term is proportional to $\delta(\mathbf{x})$ function. However, it is generated by IR rather than UV physics, which could be naively identified with $\delta(\mathbf{x})$ -like behaviour. In other words, the contact term should be interpreted as a surface integral

⁶ It is quite possible that some previously developed models such as “Vortex-Ring Model” from refs.[9, 10], might be useful in making a next step in this direction.

which is highly sensitive to the long- distances IR physics, i.e.

$$\int d^3x \delta(\mathbf{x}) \sim \int d^3x \partial_i \left(\frac{x_i}{4\pi x^3} \right) \sim \oint_S dS^i \left(\frac{x_i}{4\pi x^3} \right) \quad (59)$$

This interpretation fits nicely with microscopical picture we are advocating in this work that the corresponding contact term is generated by a long -ranged network of macroscopically large vortices in superfluid system and by the “Skeleton” in QCD as we discussed in VIII C.

If we accept this interpretation, suggesting the IR nature of the contact term (and the energy (58) associated with the contact term), one should also accept a direct consequence of such interpretation that the energy density of the system in the bulk might be highly sensitive to the boundary conditions, even though there are no any massless physical propagating degrees of freedom responsible for such sensitivity, in very much the same way as it happens in topological insulators. We reiterate this statement as follows: there is an extra energy in the bulk of the system associated with the contact terms, which however, cannot be expressed in terms of any physical propagating degrees of freedom.

In QCD context the presence of the vacuum energy not related to any physical propagating degrees of freedom was the main motivation for the proposal [38, 39] that the observed dark energy in the Universe may have, in fact, precisely such non-dispersive nature⁷. This proposal where an extra energy cannot be associated with any propagating particles should be contrasted with a commonly accepted paradigm when an extra vacuum energy in the Universe is always associated with some ad hoc propagating degree of freedom⁸.

In the superfluid context the presence of the contact term which cannot be associated with any propagating degrees of freedom implies that some observables such as superfluid density n_s are algebraically (rather than exponentially) sensitive to the boundary conditions. In other words, it is naturally to expect that the superfluid density $n_s(L)$ in a container with typical size L is slightly different from $n_s(L = \infty)$, i.e.

$$n_s(L) = n_s(L = \infty) \left[1 + \mathcal{O} \left(\frac{1}{L} \right) \right], \quad (60)$$

⁷ This novel type of vacuum energy which can not be expressed in terms of propagating degrees of freedom has in fact been well studied in QCD lattice simulations, see [38] with a large number of references on the original lattice results.

⁸ There are two instances in the evolution of the Universe when the vacuum energy plays a crucial role. The first instance is identified with the inflationary epoch when the Hubble constant H was almost constant, which corresponds to the de Sitter type behaviour $a(t) \sim \exp(Ht)$ with exponential growth of the size $a(t)$ of the Universe. The second instance where the vacuum energy plays a dominant role corresponds to the present epoch when the vacuum energy is identified with the so-called dark energy ρ_{DE} which constitutes almost 70% of the critical density. In the proposal [38, 39] the vacuum energy density can be estimated as $\rho_{DE} \sim H\Lambda_{QCD}^3 \sim (10^{-4}\text{eV})^4$, which is amazingly close to the observed value.

similar to computations in “deformed QCD” model [40]. It could be interpreted as a Topological Casimir Effect (TCE) when nontrivial topology may result in additional energy in the bulk of the system which cannot be expressed in terms of conventional propagating degrees of freedom, but rather is sensitive to topological features of the system [41, 42].

This effect (60) in many respects is very similar to TCE in the Maxwell theory formulated on a compact manifold with nontrivial $\pi_1[U(1)] = \mathbb{Z}$ when the extra term for vacuum energy is generated due to the presence of the winding states $|n\rangle$ and tunnelling transitions between them, see [41, 42] for the details. The nature of this extra vacuum energy is very different from conventional Casimir Effect which is generated due to the conventional fluctuations of the physical photons with two transverse polarizations between two neutral plates with trivial topology.

Furthermore, this extra energy can be also formulated [42] in terms of auxiliary topological fields which cannot be expressed in terms of the physical propagating photons with two transverse polarizations. These topological fields are, in fact, analogous to auxiliary fields $[a_k(\mathbf{x}), b_k(\mathbf{x})]$ introduced in the present work. More than that, the classification of the topological objects from refs. [41, 42] is based on $\pi_1[U(1)] = \mathbb{Z}$ which is equivalent to classification of the vortices in superfluid systems which is the main subject of the present work.

Essentially, eq. (60) suggests that one can use a superfluid system to study very deep and intriguing features of the QCD vacuum, in spite of the huge differences in nature of these systems as discussed in Section VIII B. In particular, the relevant structure in a superfluid system is a complicated net of vortices in Minkowski space-time, while in QCD it is a similar complicated net of configurations describing the tunnelling transitions in Euclidean space-time. Nevertheless, the key element in both systems, as emphasized in section VIII C, is there existence of a net which eventually generates the contact term (and the long-range sensitivity of some observables related to this contact term) in the bulk of the system.

The corresponding contributions to the energy in many systems (such as superfluid liquid (60), QCD [39], “deformed QCD” model [40], Maxwell theory on a compact manifold [41]) are fundamentally not expressible in terms of any physical propagating degrees of freedom. Rather, these terms reflect the topological features of a system.

• To conclude: In this work we presented one more example which shows amazing conceptual similarity between particle physics and a superfluid system. Our hope is that this work may generate future studies benefiting both fields.

ACKNOWLEDGMENTS

I am very thankful to Sasha Gorsky and Dima Kharzeev for the invaluable discussions (which essentially

initiated this project) during a workshop at the Simons Center for geometry and physics in May 2015. This work was supported in part by the National Science and Engineering Research Council of Canada.

Appendix A: Topological susceptibility in “deformed QCD”. The lessons for superfluid systems.

The main goal of this Appendix is to demonstrate a close analogy between the computations of $\langle I^2 \rangle$ for superfluid system presented in the main text and computations of $\langle Q^2 \rangle$ in QCD. Furthermore, we also show that the mass scale (54) which is generated in the superfluid system is very similar to the mass scale $m_{\eta'}$ which is generated in QCD. In both cases the new mass scales manifest themselves in studying very specific correlation functions: $\langle I^2 \rangle$ in superfluid systems, and $\langle Q^2 \rangle$ in QCD.

To proceed with this task we compute in this Appendix the topological susceptibility $\langle Q^2 \rangle$ in “deformed QCD” model developed in [13]. This is a weakly coupled gauge theory, but nevertheless preserves all the crucial elements of strongly interacting QCD, including confinement, non-trivial θ dependence, degeneracy of the topological sectors, etc. The topological susceptibility $\langle Q^2 \rangle$ plays a key role in resolution of the celebrated $U(1)_A$ problem [16–18] where the would be η' Goldstone boson generates its mass as a result of mixing of the Goldstone field with a topological auxiliary field characterizing the system.

One should emphasize that all the elements of this mechanism are well supported by the lattice simulations in strongly coupled QCD. However, in this Appendix we use the weakly coupled “deformed QCD” model to demonstrate how all the crucial elements work in this mechanism using analytical computations.

The plan of this Appendix is as follows. The basics of this model are reviewed in section A 1. In section A 2 we explain how the Goldstone boson generates its mass as a result of mixing of the Goldstone field with a topological auxiliary field. Finally, in section A 3 we reformulate our results to make very close connection with Veneziano ghost description. From this complimentary analysis one should be clear that the topological auxiliary fields are not present in the Hilbert space as the asymptotic states, and these auxiliary fields do not violate any fundamental principles of the theory.

1. The Model

In the deformed theory an extra “deformation” term is put into the Lagrangian as suggested in [13] in order to prevent the center symmetry breaking that characterizes the QCD phase transition between “confined” hadronic matter and “deconfined” quark-gluon plasma, thereby explicitly preventing that transition. We start with pure Yang-Mills (gluodynamics) with gauge group

$SU(N)$ on the manifold $\mathbb{R}^3 \times S^1$ with the standard action

$$S^{YM} = \int_{\mathbb{R}^3 \times S^1} d^4x \frac{1}{2g^2} \text{tr} [F_{\mu\nu}^2(x)], \quad (\text{A1})$$

and add to it a deformation action,

$$\Delta S \equiv \int_{\mathbb{R}^3} d^3x \frac{1}{L^3} P[\Omega(\mathbf{x})], \quad (\text{A2})$$

built out of the Wilson loop (Polyakov loop) wrapping the compact dimension

$$\Omega(\mathbf{x}) \equiv \mathcal{P} \left[e^{i \oint_{dx_4} A_4(\mathbf{x}, x_4)} \right]. \quad (\text{A3})$$

The parameter L here is the length of the compactified dimension which is assumed to be small. The coefficients of the polynomial $P[\Omega(\mathbf{x})]$ can be suitably chosen such that the deformation potential (A2) forces unbroken symmetry at any compactification scales. At small compactification L the gauge coupling is small so that the semiclassical computations are under complete theoretical control [13]. The proper infrared description of the theory is a dilute gas of N types of monopoles, characterized by their magnetic charges, which are proportional to the simple roots and affine root $\alpha_a \in \Delta_{\text{aff}}$ of the Lie algebra for the gauge group $U(1)^N$. For a fundamental monopole with magnetic charge $\alpha_a \in \Delta_{\text{aff}}$ (the affine root system), the topological charge is given by

$$Q = \int_{\mathbb{R}^3 \times S^1} d^4x \frac{1}{16\pi^2} \text{tr} [F_{\mu\nu} \tilde{F}^{\mu\nu}] = \pm \frac{1}{N}, \quad (\text{A4})$$

and the Yang-Mills action is given by

$$S_{YM} = \int_{\mathbb{R}^3 \times S^1} d^4x \frac{1}{2g^2} \text{tr} [F_{\mu\nu}^2] = \frac{8\pi^2}{g^2} |Q|. \quad (\text{A5})$$

The θ -parameter in the Yang-Mills action can be included in conventional way,

$$S_{YM} \rightarrow S_{YM} + i\theta \int_{\mathbb{R}^3 \times S^1} d^4x \frac{1}{16\pi^2} \text{tr} [F_{\mu\nu} \tilde{F}^{\mu\nu}], \quad (\text{A6})$$

with $\tilde{F}^{\mu\nu} \equiv \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$.

The system of interacting monopoles, including the θ parameter, can be represented in the sine-Gordon form as follows [13, 14],

$$S_{\text{dual}} = \int_{\mathbb{R}^3} d^3x \frac{1}{2L} \left(\frac{g}{2\pi} \right)^2 (\nabla \boldsymbol{\sigma})^2 - \zeta \int_{\mathbb{R}^3} d^3x \sum_{a=1}^N \cos \left(\alpha_a \cdot \boldsymbol{\sigma} + \frac{\theta}{N} \right), \quad (\text{A7})$$

where ζ is magnetic monopole fugacity which can be explicitly computed in this model using the conventional semiclassical approximation. The θ parameter enters the effective Lagrangian (A7) as θ/N which is the direct consequence of the fractional topological charges of the monopoles (A4). Nevertheless, the theory is still 2π periodic. This 2π periodicity of the theory is restored not

due to the 2π periodicity of Lagrangian (A7). Rather, it is restored as a result of summation over all branches of the theory when the levels cross at $\theta = \pi(\text{mod } 2\pi)$ and one branch replaces another and becomes the lowest energy state as discussed in [14].

Finally, the dimensional parameter which governs the dynamics of the problem is the Debye correlation length of the monopole's gas,

$$m_\sigma^2 \equiv L\zeta \left(\frac{4\pi}{g} \right)^2. \quad (\text{A8})$$

The average number of monopoles in a ‘‘Debye volume’’ is given by

$$\mathcal{N} \equiv m_\sigma^{-3} \zeta = \left(\frac{g}{4\pi} \right)^3 \frac{1}{\sqrt{L^3 \zeta}} \gg 1, \quad (\text{A9})$$

The last inequality holds since the monopole fugacity is exponentially suppressed, $\zeta \sim e^{-1/g^2}$, and in fact we can view (A9) as a constraint on the region validity where semiclassical approximation is justified. This parameter \mathcal{N} is therefore one measure of ‘‘semi-classicality’’.

2. How the Goldstone field receives its mass

In the sine-Gordon formulation (A7) the η' meson field appears exclusively in combination with the θ parameter as $\theta \rightarrow \theta - \phi(x)$, where ϕ is the phase of the chiral condensate which, up to dimensional normalization parameter, is identified with physical η' meson in QCD. As it is well known, this property is the direct result of the transformation properties of the path integral measure under the chiral transformations $\psi \rightarrow \exp(i\gamma_5 \frac{\phi}{2})\psi$. Therefore, $\phi(x)$ enters the effective action (A7) exactly in the combination $(\theta - \phi(\mathbf{x}))/N$ when we include light quarks into the system.

The next step in our presentation is the computation of the topological susceptibility in this model. These computations can be explicitly carried out as the system is weakly coupled gauge theory, and the semiclassical approximation under complete theoretical control. The result of this computation is [14]:

$$\langle q(\mathbf{x}), q(\mathbf{0}) \rangle_{QCD} = \frac{\zeta}{NL^2} \left[\delta(\mathbf{x}) - m_{\eta'}^2 \frac{e^{-m_{\eta'} r}}{4\pi r} \right], \quad (\text{A10})$$

where new mass scale $m_{\eta'}$ is generated in the problem and is determined in terms of the original dimensional parameters ζ and L ,

$$m_{\eta'}^2 = \frac{\zeta}{cN}, \quad \frac{c}{L} = f_{\eta'}^2 \quad (\text{A11})$$

Originally, formula (A10) was derived in [14] by explicit integration over all possible monopole's configurations, summing over all possible their orientations in the gauge group $SU(N)$, etc. However, we want to derive this formula using a different technique in terms of the

auxiliary topological fields, similar to the $[a_k(\mathbf{x}), b_k(\mathbf{x})]$ introduced in section VIA. Such a computation would demonstrate a close connection with the analysis of the winding number correlation function $\langle I^2 \rangle$ computed in section VIB.

In fact, the corresponding computations have been already carried out in [15], and we present the relevant formulae in this Appendix. The basic technical idea is exactly as presented in sections VIA, VIB when the auxiliary topological fields (treated as the slow degrees of freedom) are introduced into the system by inserting the functional δ function similar to eq. (31). The next step is to integrate out the fast degrees of freedom in the background of slow auxiliary topological fields. This model is a weakly coupled gauge theory. Therefore, these computations can be carried out exactly. The result is [15]:

$$\begin{aligned} \mathcal{Z} &\sim \int \mathcal{D}[b] \mathcal{D}[\sigma] \mathcal{D}[a] \mathcal{D}[\phi] e^{-(S_{\text{top}} + S_{\text{dual}}[\sigma, b, \phi] + S_\phi)} \quad (\text{A12}) \\ S_\phi &= \int_{\mathbb{R}^3} d^3x \frac{c}{2} (\nabla \phi)^2 \\ S_{\text{top}}[b, a] &= \frac{-i}{4\pi N} \int_{\mathbb{R}^3} d^3x b(\mathbf{x}) \vec{\nabla}^2 a(\mathbf{x}); \\ S_{\text{dual}}[\sigma, b, \phi] &= \int_{\mathbb{R}^3} d^3x \frac{1}{2L} \left(\frac{g}{2\pi} \right)^2 (\nabla \sigma)^2 \\ &\quad - \zeta \int_{\mathbb{R}^3} d^3x \sum_{a=1}^N \cos \left(\alpha_a \cdot \sigma + \frac{\theta + b(\mathbf{x}) - \phi(\mathbf{x})}{N} \right), \end{aligned}$$

This formula is analogous to expression (38) derived for superfluid system. However, there is a fundamental difference between these two computations. Formula (A12) is an exact result of computations in semiclassical approximation in a weakly coupled gauge theory. Therefore, all terms, including the interaction terms are accounted in (A12).

It should be contrasted with formula (38) derived for superfluid system. In that case we did not sum over all possible configurations, including all geometries and topologies of a complicated network of knotted, folded and wrinkled vortices. Instead, we used a gauge invariance arguments to restore the structure of the quadratic portion of the Hamiltonian (38). We neglected all the interactions in partition function (38), see few comments on this simplifications at the end of section VIB. Another difference between (A12) and (38) is that the auxiliary fields $[b(\mathbf{x}), a(\mathbf{x})]$ in the deformed QCD model are scalars, while in superfluid system the auxiliary fields $[b_k(\mathbf{x}), a_k(\mathbf{x})]$ are vectors. The difference can be traced from the fact that the relevant objects in the deformed QCD model are point-like monopoles, while in superfluid systems the relevant objects are the vortices which are classified by a specific direction of the circulation field $\gamma_k(\mathbf{x})$.

We now return to analysis of the deformed QCD model. Our task now is to compute the topological susceptibility using the partition function (A12). The effective action

assumes the following form

$$\begin{aligned} \langle q(\mathbf{x}), q(\mathbf{0}) \rangle_{QCD} &= \frac{1}{\mathcal{Z}} \int \frac{\mathcal{D}[a] e^{-S_{QCD}} [\vec{\nabla}^2 a(\mathbf{x}), \vec{\nabla}^2 a(\mathbf{0})]}{(4\pi N L)^2} \\ S_{QCD}[a, \phi] &= \frac{1}{2N\zeta(4\pi)^2} \int_{\mathbb{R}^3} d^3x \left[a(\mathbf{x}) \vec{\nabla}^2 \vec{\nabla}^2 a(\mathbf{x}) \right] \\ &+ \int_{\mathbb{R}^3} d^3x \left[\frac{c}{2} \left(\vec{\nabla} \phi(\mathbf{x}) \right)^2 + \frac{i}{4\pi N} \vec{\nabla} \phi(\mathbf{x}) \cdot \vec{\nabla} a(\mathbf{x}) \right], \quad (\text{A13}) \end{aligned}$$

which should be compared with similar expressions (41), (42) derived for superfluid system.

The next step in computation of the topological susceptibility is the diagonalization of the action to eliminate the non-diagonal term $\int d^3x \vec{\nabla} \phi \cdot \vec{\nabla} a$ in (A13) by making a shift

$$\frac{\phi_2(\mathbf{x})}{\sqrt{c}} \equiv \phi(\mathbf{x}) + \frac{i}{4\pi c N} a(\mathbf{x}). \quad (\text{A14})$$

The problem is reduced to the computations of the Gaussian integral with the effective action (after the rescaling), taking the form

$$\begin{aligned} S_{QCD}[a', \phi_2] &= \frac{1}{2} \int_{\mathbb{R}^3} d^3x \left(\vec{\nabla} \phi_2(\mathbf{x}) \right)^2 \\ &+ \frac{1}{2} \int_{\mathbb{R}^3} d^3x a'(\mathbf{x}) \left[\vec{\nabla}^2 \vec{\nabla}^2 - m_{\eta'}^2 \vec{\nabla}^2 \right] a'(\mathbf{x}) \end{aligned} \quad (\text{A15})$$

which plays the same role as the Hamiltonian (48) in our computations for superfluid system. In formula (A15) parameter $m_{\eta'}^2$ is the η' mass in this model and it is given by eq.(A11). In terms of this rescaled field $a'(\mathbf{x})$ the Gaussian integral which enters (A13) can be easily computed and it is given by

$$\frac{\int \mathcal{D}[a'] e^{-S_{QCD}} \vec{\nabla}^2 a'(\mathbf{x}), \vec{\nabla}^2 a'(\mathbf{0})}{\int \mathcal{D}[a'] e^{-S_{QCD}[a']}} = \left[\delta(\mathbf{x}) - m_{\eta'}^2 G_{m_{\eta'}}(\mathbf{x}) \right]$$

where S_{QCD} is defined by (A15) and the massive Green's function $G_{m_{\eta'}}(\mathbf{x}) = \frac{e^{-m_{\eta'} r}}{4\pi r}$ is normalized in conventional way ($m_{\eta'}^2 \int d^3x G_{m_{\eta'}}(\mathbf{x}) = 1$). Collecting all numerical coefficients from (A13) and (A16) the final expression for the topological susceptibility in the presence of massless quark takes the form

$$\langle q(\mathbf{x}), q(\mathbf{0}) \rangle_{QCD} = \frac{\zeta}{NL^2} \left[\delta(\mathbf{x}) - m_{\eta'}^2 \frac{e^{-m_{\eta'} r}}{4\pi r} \right]. \quad (\text{A16})$$

This precisely reproduces our previous formula (A10) which was derived by explicit integration over all possible monopole's configurations without even mentioning the topological auxiliary fields. The celebrated $U(1)_A$ problem is resolved in this framework exclusively as a result of dynamics of the topological $a(\mathbf{x}), b(\mathbf{x})$ fields. These fields are not propagating degrees of freedom, but nevertheless generate a crucial non-dispersive contribution with the “wrong sign” which is the key element for the formulation and resolution of the $U(1)_A$ problem and the generation of the η' mass. Formula (A16) has precisely

the same structure as eq. (50) in our studies on superfluidity.

The main lesson for our present studies of the superfluid system developed in sections VIA, VIB, VIC is as follows. The key correlation function (A10), (A16) can be computed using two different techniques. First, one can use an explicit direct computation which requires an explicit summation and integration over all possible monopole's configurations, including positions and orientations, to arrive to (A10). This is straightforward but technically very involved procedure. The second technique uses the auxiliary topological fields which effectively account for the long range dynamics of the monopole's configurations. The corresponding formula (A16) exactly reproduces the direct computations (A10). The lesson is: this procedure with auxiliary fields serves as a test and gives us a confidence that this formal manipulations with the auxiliary fields reproduces the correct results.

• In our present studies of superfluid system we do not have a proper technique capable for direct computations of the winding number susceptibility $\langle I^2 \rangle$ by summing over all possible vortex configurations including all geometries and topologies of a complicated network of knotted, folded and wrinkled vortices. Therefore, we used in sections VIA, VIB a second technique which includes the auxiliary topological fields. We tested this technique in the “deformed QCD” model. Therefore, we are quite confident in qualitative picture developed in sections VIA, VIB, including the generation of the new mass scale (48), (54) and the contact term (50).

3. Topological fields and the Veneziano ghost.

The main goal of this subsection is to argue that unphysical pole (57) which was found in our studies in section VIC is absolutely harmless and it does not correspond to any unphysical or ghost-like behaviour in the system. We use the “deformed QCD” model (where all computations can be carried out exactly) to support our claim. In addition, the expression for the correlation function (A16) with action (A15) can be represented in a complementary way which makes the connection between the Veneziano ghost and topological fields much more explicit and precise.

To proceed with our task, we use a standard trick to represent the 4-th order operator $[\vec{\nabla}^2 \vec{\nabla}^2 - m_{\eta'}^2 \vec{\nabla}^2]$ which enters the effective action (A15) as a combination of two terms with the opposite signs: a ghost field ϕ_1 and a massive physical $\hat{\phi}$ field. To be more specific, we write

$$\frac{1}{[\vec{\nabla}^2 \vec{\nabla}^2 - m_{\eta'}^2 \vec{\nabla}^2]} = \frac{1}{m_{\eta'}^2} \left(\frac{1}{\vec{\nabla}^2 - m_{\eta'}^2} - \frac{1}{\vec{\nabla}^2} \right), \quad (\text{A17})$$

such that the Green's function for the $a(\mathbf{x})$ field which enters the expression for the topological susceptibility

(A16) can be represented as a combination of two Green's functions, for the physical massive field with conventional kinetic term and for the ghost field with the “wrong” sign for the kinetic term. This formula is identically the same as (49) which was employed in the main text on superfluidity.

Naively, the presence of 4-th order operator in eq. (A15) is a signal that the ghost is present in the system. This signal is explicit in eq. (A17). The contact term in this framework is represented by the ghost contribution. Indeed, the relevant correlation function which enters the expression for the topological susceptibility (A16) can be explicitly computed using expression (A17) for the inverse operator as follows

$$\begin{aligned} & \frac{\int \mathcal{D}[a'] e^{-S_{QCD}[a']} \vec{\nabla}^2 a'(\mathbf{x}), \vec{\nabla}^2 a'(\mathbf{0})}{\int \mathcal{D}[a'] e^{-S_{QCD}[a']}} \\ &= \int \frac{d^3 p}{(2\pi)^3} e^{-ipx} \frac{p^4}{m_{\eta'}^2} \left[-\frac{1}{p^2 + m_{\eta'}^2} + \frac{1}{p^2} \right] \\ &= \int \frac{d^3 p}{(2\pi)^3} e^{-ipx} \left[\frac{p^2}{p^2 + m_{\eta'}^2} \right] = \left[\delta(\mathbf{x}) - m_{\eta'}^2 \frac{e^{-m_{\eta'} r}}{4\pi r} \right], \end{aligned} \quad (\text{A18})$$

which, of course, is the same final expression we had before (A16) with the only difference being that it is now explicitly expressed as a combination of two terms: a physical massive η' contribution and an unphysical contribution which saturates the contact term with the “wrong” sign.

Such interpretation can be supported by computing $\langle a'(\mathbf{x}), a'(\mathbf{0}) \rangle$ itself, similar to our studies of the correlation function (57).

$$\begin{aligned} \langle a'(\mathbf{x}), a'(\mathbf{0}) \rangle &= \frac{\int \mathcal{D}[a'] e^{-S_{QCD}[a']} a'(\mathbf{x}), a'(\mathbf{0})}{\int \mathcal{D}[a'] e^{-S_{QCD}[a']}} \\ &= \int \frac{d^3 p}{(2\pi)^3} e^{-ipx} \frac{1}{m_{\eta'}^2} \left[-\frac{1}{p^2 + m_{\eta'}^2} + \frac{1}{p^2} \right] \end{aligned} \quad (\text{A19})$$

Naively, the presence of the pole at $p^2 = 0$ with a “wrong sign” in eq. (A19) is a signal of a ghost in the system, which implies the violation of unitarity along with other fundamental principles of quantum field theory (QFT). Nevertheless, we know that the original theory is perfectly defined QFT, and the generation of this unphysical pole is simply an artifact of our formal procedure, when we inserted auxiliary topological fields into the system. Indeed, we observed above that one computes a gauge invariant correlation function (A18) this “wrong sign” contribution generates the contact term, and it does not correspond to any propagation of any physical degrees of freedom. Another way to support this claim is to construct the physical Hilbert space for the problem, which is our next exercise.

To proceed with this task we represent the correlation function (A18) by introducing two fields $\phi_1(\mathbf{x})$ and $\hat{\phi}(\mathbf{x})$ replacing the $a'(\mathbf{x})$ which enters the effective action (A15) as the 4-th order operator. To be more precise, we rewrite

our action (A15) in terms of these new fields $\phi_1(\mathbf{x})$ and $\hat{\phi}(\mathbf{x})$ as follows

$$\begin{aligned} S_{QCD}[\hat{\phi}, \phi_1, \phi_2] &= \frac{1}{2} \int_{\mathbb{R}^3} d^3 x \left[\left(\vec{\nabla} \phi_2(\mathbf{x}) \right)^2 - \left(\vec{\nabla} \phi_1(\mathbf{x}) \right)^2 \right] \\ &+ \frac{1}{2} \int_{\mathbb{R}^3} d^3 x \left[\left(\vec{\nabla} \hat{\phi}(\mathbf{x}) \right)^2 + m_{\eta'}^2 \hat{\phi}^2(\mathbf{x}) \right] \end{aligned} \quad (\text{A20})$$

with the $a'(\mathbf{x})$ field expressed in terms of the new fields $\phi_1(\mathbf{x})$ and $\hat{\phi}(\mathbf{x})$ as

$$a'(\mathbf{x}) \equiv \frac{1}{m_{\eta'}} \left(\hat{\phi}(\mathbf{x}) - \phi_1(\mathbf{x}) \right), \quad (\text{A21})$$

while the topological density $q(\mathbf{x})$ operator is expressed in terms of these fields as

$$q = \sqrt{\frac{\zeta}{NL^2}} \vec{\nabla}^2 a' = \sqrt{\frac{\zeta}{NL^2 m_{\eta'}^2}} \vec{\nabla}^2 (\hat{\phi} - \phi_1). \quad (\text{A22})$$

This redefinition obviously leads to our previous result (A16), (A18) when we use the Green's functions determined by the Lagrangian (A20) for the physical massive field $\hat{\phi}$ and the ghost ϕ_1 ,

$$\langle q(\mathbf{x}), q(\mathbf{0}) \rangle_{QCD} = \frac{\zeta}{NL^2} \left[\delta(\mathbf{x}) - m_{\eta'}^2 \frac{e^{-m_{\eta'} r}}{4\pi r} \right] \quad (\text{A23})$$

An important point here is that the contact term in this framework is explicitly saturated by the topological non-propagating auxiliary fields expressed in terms of the ghost field ϕ_1 , similar to the Kogut-Susskind (KS) ghost [19] in two dimensional QED, or the Veneziano ghost [17] in four-dimensional QCD. From our original formulation [14] without any auxiliary fields it is quite obvious that our theory is unitary and causal. When we introduce the auxiliary fields (which are extremely useful when one attempts to study the long range order) the unitarity, of course, still holds. Formally, the unitarity holds in this formulation because the ghost field ϕ_1 is always paired up with ϕ_2 in every gauge invariant matrix element as explained in [19] (with the only exception being the topological density operator (A22) which requires a special treatment presented in this section). The condition that enforces this statement is the Gupta-Bleuler-like condition on the physical Hilbert space $\mathcal{H}_{\text{phys}}$ which reads

$$(\phi_2 - \phi_1)^{(+)} | \mathcal{H}_{\text{phys}} \rangle = 0, \quad (\text{A24})$$

where the (+) stands for the positive frequency Fourier components of the quantized fields.

•The crucial point here is that the formulation of the theory using the topological fields has an enormous advantage as the long range dynamics is explicitly accounted for in the formulation (A12) and therefore, in the equivalent formulation in terms of the ghost field (A20). In solvable “deformed QCD” model it was a question of taste which framework to choose. For our studies of superfluidity developed in sections VI, VII this is not a

question of taste, but necessity. This is because an explicit studies of the vortex network which require an accounting for all the configurations with varies geometries and topologies of knotted, wrinkled and folded vortices is simply not technically feasible. At the same time, the studies with the topological auxiliary fields give, at least, qualitative picture of the system. The computations in “deformed QCD” model (where the explicit computa-

tions have been carried out) presented in this Appendix give us some confidence that our formal manipulations with the path integral with topological auxiliary fields in sections VI, VII capture the basic features of the system.

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